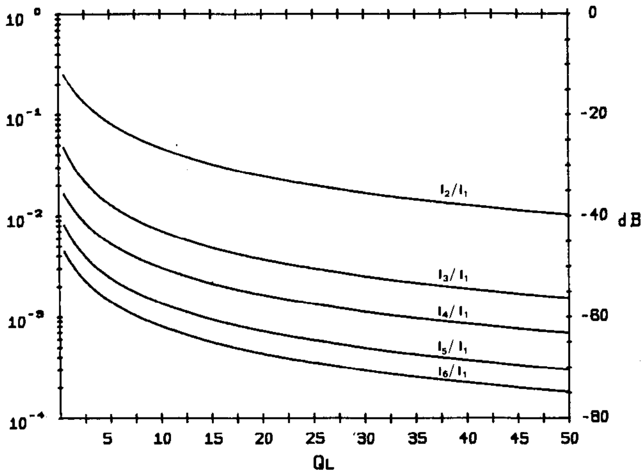

Fig. 9. Harmonic content of the output current as a function of Q_L.

values given here are in very good agreement with those given in [4].

V. Conclusions

A new method for the analysis and design of the class E tuned power amplifier taking into account the Q_L factor has been described and implemented for the special case of the series-tuned, shunt-capacitance load-network configuration.

Load-network design element values were obtained as a function of Q_L for optimum amplifier performance. These values were used to plot curves for the device stress, the power output, and the power output capability as functions of Q_L. These curves were compared with the corresponding curves obtained using element values given by Sokal [3] and Raab [2]. Curves were also plotted which show the influence of the shunting capacitance and the detuning portion of L_2 on the design requirements for various values of Q_L. Finally, the harmonic content of the output current was obtained and plotted as a function of Q_L.

The method presented here can be used for the analysis and design of other class E amplifier configurations [6], or with more complicated circuits in practical designs. Such an analysis might prove very useful because it ensures optimum amplifier performance for all values of Q_L, it provides the designer with quantitative criteria for the usual tradeoffs concerning Q_L, and it makes the amplifier performance more predictable and the design more reliable.

The design and performance values obtained in this paper have been tested using computer simulation [8] as well as laboratory experiments. The experimental results agree very closely, especially when the circuit conditions were made to approximate closely the assumptions made.

Acknowledgment

The authors wish to express their appreciation to the anonymous reviewers for their useful comments and suggestions.

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The Stochastic Process of Transitions between Limit Cycles for a Special Class of Self-Oscillators under Random Perturbations

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Abstract **—A plenary investigation of the stochastic transitions between neighborhoods of limit cycles in a randomly perturbed special class of oscillatory circuits is presented. The very possibility of the realization of self-oscillators with a given set of several limit cycles was established in the recent works of the authors. On this ground, a simple analysis of the associated diffusion process and its properties are given.**

I. Introduction

Interest in the behavior of dynamical systems with multiple steady states and under random perturbation appeared in the early 1930's [1]. Stochastic processes of the diffusion type originating in this way have been receiving increasingly broad interpretations (see [2], [3] and references therein), while particular attention to systems with several limit cycles in applied physics, electronics, and chemical kinetics is focused in [1], [4]–[6], and [15].

Most investigations in the field are based on purely theoretical considerations or hypothetical models with no requisite control over the dynamical processes in the concrete physical object. In analytical studies, as a rule, serious difficulties arise when attempting to solve the associated Fokker–Planck equations, even approximately. Therefore, any concrete example with an explicitly solvable Fokker–Planck equation is of significant value. A paradigm of certain changes in commonly known oscillatory dynamical systems, providing a basis for complete stochastic analysis, can be found in the works of Caughey [7], [8]. The "distorted" system, besides being interesting on its own, preserves the qualitative phase-space portrait of the original one and therefore they both behave similarly under random perturbation.

Thus, we work with a dynamical system which does not need any use of "small perturbation" theories. Our oscillator is strongly nonlinear (not quasi-linear), and its limit cycles are determined exactly, not through approximations. Moreover, the solution to the Fokker–Planck equation is also *exact* and is not, in any direct way, related to the asymptotic theory of small random

Manuscript received March 13, 1986; revised October 14, 1986.

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IEEE Log Number 8714323.

perturbations in quasi-linear stochastic "systems with averaging" [16]. Nevertheless, in our analysis, we make use of the asymptotic estimates of the exact solution to the Fokker–Planck equation with respect to a parameter which ordinarily appears in problems for elliptic equations with singular perturbation (see, e.g., [17]).

Although, as mentioned above, the discussions regarding the character of the described diffusion processes began over 50 years ago [1], this work represents their first explicit experimental observation.

II. Deterministic System With Multiple Cycles

In a framework of quasi-linear theory, the simplest way of arriving at a system with several limit cycles would be a Lienard equation:

$$\frac{d^2x}{dt^2} + \mu W(x)\frac{dx}{dt} + x = 0. \tag{1}$$

The most natural generalization of the Van der Pol equation occurs for $W(x)$ in (1), such that (see [9], [10], [14])

$$W(x) = \prod_{k=1}^{N} \left(x^2 - a_k^2\right).$$

However, contrary to the Van der Pol equation when $N=1$, such a choice of $W(x)$ does not guarantee the appearance of N limit cycles; nor does it provide for an easily solvable Fokker–Planck equation. In a very recent work [9], for example, both problems are alleviated by preliminary approximate asymptotic assumptions and informal heuristic reasoning. It is stated there that for $N=3$, "once the amplitudes of the stable cycles are fixed, they uniquely determine the amplitude of the unstable cycle…." Obviously, in general, this statement is incorrect [10]. We will use a slightly different model:

$$\frac{d^2x}{dt^2} + \mu\left[\prod_{k=1}^{N}\left(2Z - a_k^2\right)\right]\frac{dx}{dt} + x = 0 \tag{2}$$

where

$$Z \equiv \frac{1}{2}\left(\dot{x}^2 + x^2\right), \qquad \mu > 0.$$

The system in (2) has N limit cycles corresponding to periodical solutions $x_k = a_k \sin t$. If $a_1 > a_2 > \cdots > a_N > 0$, then all limit cycles with odd number k are orbitally asymptotically stable, and those for even k are unstable. As for the equilibrium $x \equiv 0$, it is unstable for odd N and asymptotically stable for even N. Thus, we arrive at a situation as described in [16, ch. 7, sec. 1]. Considering the oscillator with one degree of freedom, the authors [16] single out conditions for the existence of more than one limit cycle in terms of Van der Pol variables. An essential condition for their approach is the presence of the small parameter at the nonlinear terms in the equation of the oscillator. We emphasize that the parameter μ in (2) may assume arbitrary values, unlike the assumptions of [16]; i.e., the periodical solutions of oscillator (2) are *totally independent* of the value of parameter μ. Because of this circumstance, the Lyapunov exponents of periodical solutions to (2), being proportional to μ, can assume arbitrarily large values. This possibility does not exist in quasi-linear systems (these systems can have only "small" Lyapunov exponents) studied in [16]–[18].

III. A Stochastically Driven Oscillator

When oscillator (2) is driven with white noise,

$$\frac{d^2x}{dt^2} + \mu\left[\prod_{k=1}^{N}\left(2Z - a_k^2\right)\right]\frac{dx}{dt} + x = (2\sigma)^{1/2}\xi(t).$$

where $\sigma = \text{constant} > 0$ and $\xi(t)$ is a "white noise" with

$$\langle \xi(t)\rangle = 0 \quad \langle \xi(t)\xi(\tau)\rangle = \delta(t-\tau),$$

The stationary probability density $p(x, dx/dt)$ satisfies the following Kolmogorov–Fokker–Planck equation:

$$0 = -x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left[\mu x_2 \prod_{k=1}^{N}\left(2h - a_k^2\right) + x_1\right]p\right\} + \sigma^2\frac{\partial^2 p}{\partial x_2^2}$$

in which

$$x_1 \equiv x, x_2 \equiv \frac{dx}{dt} \quad \text{and} \quad h \equiv \frac{1}{2}\left(x_1^2 + x_2^2\right). \tag{3}$$

Due to [3] and [8], the Green function in the space R^{2N} for the degenerate elliptic equation (3) is

$$p(x_1, x_2) = \exp\left(-\frac{\mu}{\sigma}S_N(h)\right) \tag{4}$$

where

$$S_N(h) = \int_0^h \prod_{k=1}^{N}\left(2h - a_k^2\right) dh.$$

By introducing the norming factor

$$C_N = 2\pi \int_{-\infty}^{\infty} \exp(-\mu S_N(h))\, dh$$

where $\gamma \equiv \mu/\sigma$, the solution to (3) can be represented as

$$p_N(x_1, x_2) = C_N \exp[-\gamma S_N(h)].$$

Consider now a few particular cases. When $N=2$, the so called hard self-excitation [13] occurs, and the equilibrium $(x_1, x_2) = (0,0)$ coexists with the two limit cycles:

$$x_1^2 + x_2^2 = a_k^2 \qquad (k=1,2) \qquad a_1 > a_2$$

which are stable and unstable, respectively.

For the probabilities $\Psi(D_0), \Psi(D_1)$ of locations within the regions:

$$D_0 = \left\{(x_1, x_2)\colon 2h \leqslant \epsilon_0^2\right\}$$

$$D_1 = \left\{(x_1, x_2)\colon (a_1 - \epsilon_1)^2 \leqslant 2h \leqslant (a_1 + \epsilon_1)^2\right\}, \qquad \epsilon_1 > 0$$

respectively, we obtain

$$\Psi_2(D_0) = 2\pi C_2 \int_0^{w_1} e^{-\gamma S_2(t)}\, dt \tag{5}$$

$$\Psi_2(D_1) = 2\pi C_2 \int_{w_2}^{w_3} e^{-\gamma S_2(t)}\, dt \tag{6}$$

where $w_1 = \epsilon_1^2/2$, $w_2 = (a_1 - \epsilon_2)^2/2$, and $w_3 = (a_1 + \epsilon_2)^2/2$. The concept of transitions between limit cycles implies that the time spent in the neighborhood of each limit cycle is significantly greater than the time spent outside the neighborhoods. For this reason, we shall resort to asymptotic estimates for "sufficiently large" γ. Let T be a set:

$$T = \left\{R^2\colon (x_1, x_2)\right\}/D_0 \cup D_1.$$

Now, note that the function $S_2(t)$ has two minima at the points $t=0$ and $t = a_1^2$. Therefore, applying asymptotic estimates [11], [12] for the probability $\Psi_2(T)$ yields

$$\Psi_2(T) \sim 0$$

when $\gamma \to \infty$.

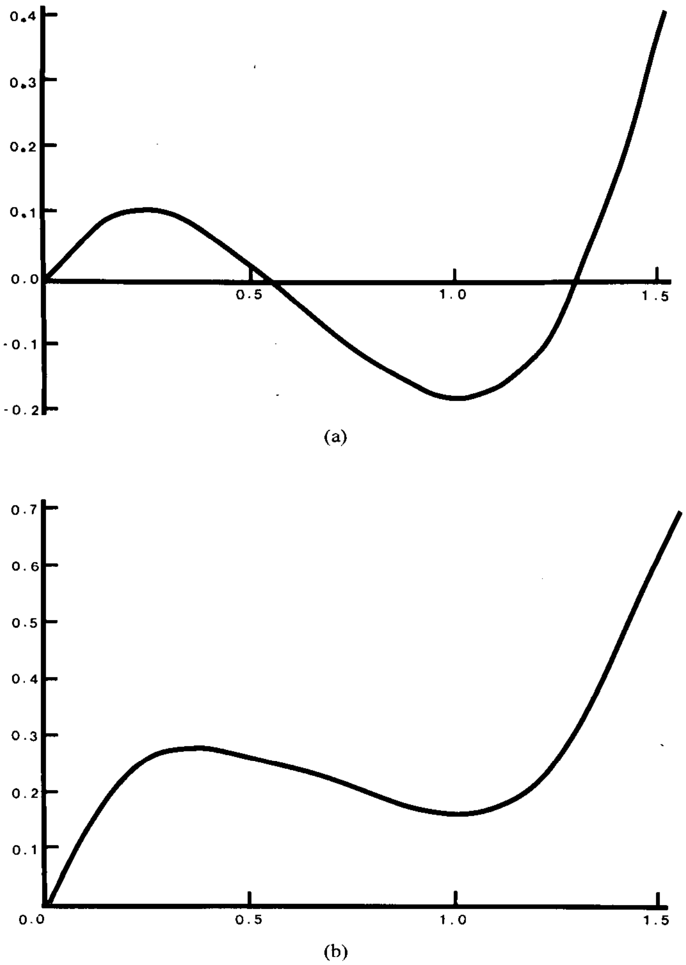

Fig. 1. Illustrative $S_2(h)$ for (a) a fully stable limit cycle and (b) a fully stable origin.

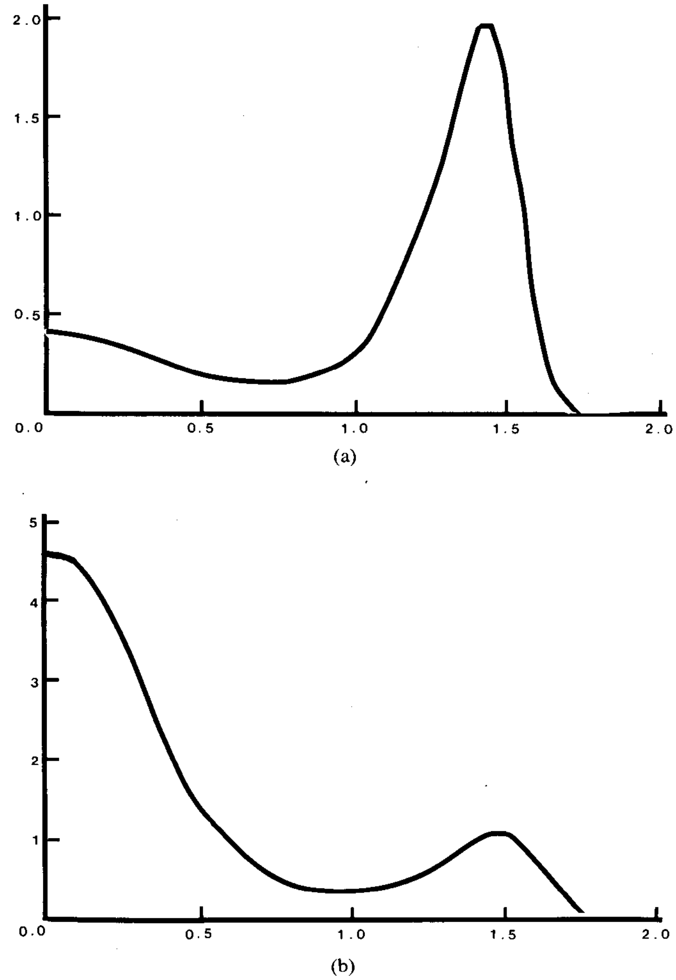

Fig. 2. $p_2(x)$ for $\gamma = 10$, $a_1^2 = 2$. (a) $a_2^2 = 0.49$. (b) $a_2^2 = 0.81$.

Using the Laplace method for asymptotic estimates of integrals in (5) and (6), one obtains [12, ch. 1]

$$K(\gamma) \equiv \frac{\Psi_2(D_0)}{\Psi_2(D_1)} \sim \frac{e^{\gamma S_2(a_1^2)}\left(a_1^2 - a_2^2\right)^{1/2}}{\pi a_1^2 a_2^2 \gamma^{1/2}}. \tag{7}$$

Following standard terminology, the point at which $S_N(h)$ attains its global minimum value, on the interval $[0, +\infty]$, is called a "fully" stable state (authors of [16, ch. 7, sect. 8, example 3] call such a limit cycle the "most stable" one), while other local minima are termed metastable states. In this regard, the stable equilibrium (stable limit cycle) is a metastable state with respect to a small white noise perturbation depending on whether the situation in Fig. 1(a) or (b) is taking place. In other words, a long-lasting noise perturbation "pushes" the oscillator to an arbitrarily small neighborhood of the stable state associated with the absolute minimum of $S_2(h)$. It is of physical significance that an oscillator with hard self-excitation, under small noise perturbation, can "avoid" the stable limit cycle.

Resorting to the situation with two stable limit cycles, $N = 3$, one should notice that the only possible transitions can occur between limit cycles with amplitudes a_1 and a_3, since the origin and the cycle with amplitude a_2 are unstable. The above-described solutions for the case $N = 2$ can be easily extended for two stable limit cycles and more. Separating stable and metastable limit cycles requires determination of the global and local minima of $S_3(h)$. Additional information will be provided in the next section.

Remark: Out of a large body of literature (e.g., [16]–[19]) related to the problem of random perturbations of dynamical systems, only [16, ch. 7, sec. 8] mentions a situation with several limit cycles. The method used in [16] is essentially based on the fact that the dynamical system is quasi-linear *and* the random perturbation is "small." The authors of [16] make only qualitative conclusions relying on a quasi-potential and action functional introduced in [16, chs. 2 and 3]. Their construction coincides with the exact solution based on the function S_N (in this paper). At the end of [16, ch. 7, sec. 1], one finds the phrase, "...using such approximations we can make a number of interesting conclusions" (on the behavior of the oscillator with several cycles). However, to obtain the same qualitative and quantitative results in their concrete form, we did not need to resort to the assumptions or to the analytical machinery used in [16].

IV. Experiments and Some Numerical Illustrations

All the experimental and numerical results which follow are attributed to the oscillator governed by (2) only. An important question in regard to this paper is how large the parameter γ must be for the asymptotic estimates to be appropriate.

For concrete numerical illustrations of the qualitative effects of variations in γ and N described in Section III, we resort to examples for the cases of $N = 2$, 3, and 5. For $N = 2$, Fig. 2(a)

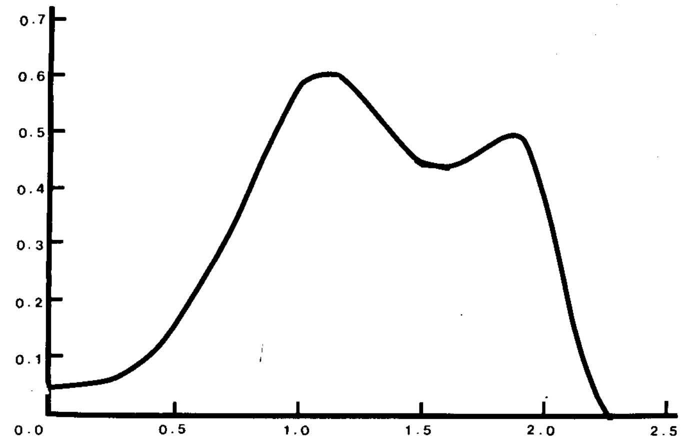

Fig. 3. $p_3(x)$ for $\gamma=1$, $a_1^2=1.88$, $a_2^2=1.6$, and $a_3^2=1.1$.

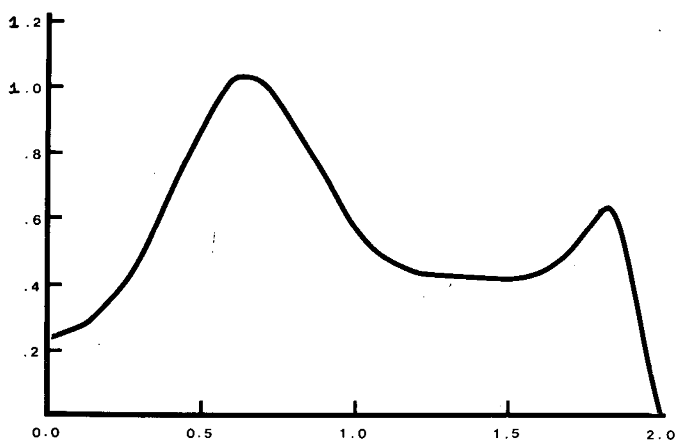

Fig. 4. $p_5(x)$ for $\gamma=2$, $a_1^2=3.3$, $a_2^2=2.33$, $a_3^2=1.93$, $a_4^2=1.61$, $a_5^2=0.403$.

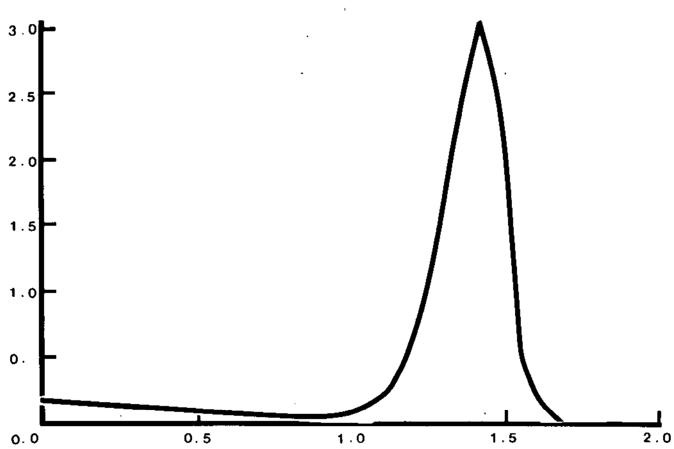

Fig. 5. $p_2(x)$ for $\gamma=20$, $a_1^2=2$, $a_2^2=0.49$.

and (b) demonstrates the transition of the metastable state to the fully stable state depending on the location of the unstable cycle. Figs. 3 and 4 present Ψ_N for $N=3$ and 5, respectively. As seen in Fig. 5, asymptotic estimates are in good agreement with the exact solution for $\gamma=20$, as indicated by the shape of $p_2(x)$.

The above-described class of nonlinear oscillators is easily realizable as compact electronic circuits. The block diagram for the noise-excited oscillator is shown in Fig. 6. It is interesting to note that only a few years ago, the possibility of realizing such hardware in compact form did not exist. The group of experiments was performed for $N=2$, 3, and 5. Figs. 7, 8, and 9

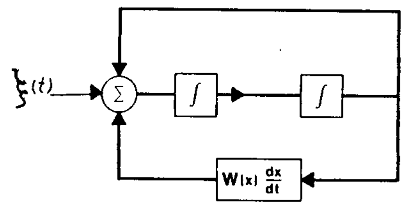

Fig. 6. Block diagram for experimental oscillator built from readily available components.

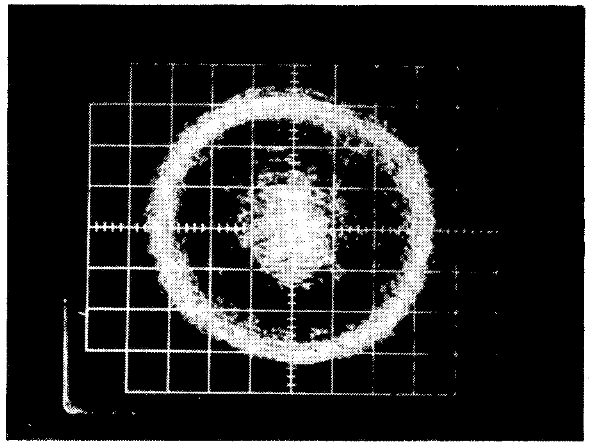

Fig. 7. Stochastic transitions in the phase plane as measured on an oscilloscope for the circuit of Fig. 6, $\gamma=2$. Scale setting was 0.5 V/div for $N=2$ with $a_1^2=2.28$ V, $a_2^2=0.6$ V.

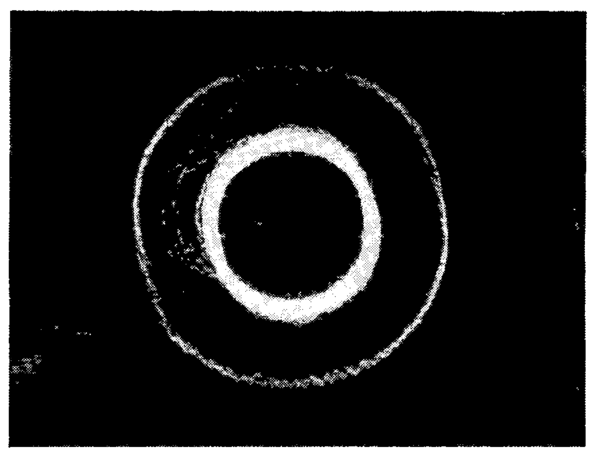

Fig. 8. Stochastic transitions in the phase plane as measured on an oscilloscope for the circuit of Fig. 6, $\gamma=2$. Scale setting was 0.5 V/div for $N=3$ with $a_1^2=1.22$ V, $a_2^2=2.2$ V, $a_3^2=3.52$ V.

illustrate the stochastic dynamics of transitions between limit cycles. For sufficiently large γ, the diffusion process is reduced to a Markov process with a finite discrete set of states. From these single-exposure photographs, the neighborhoods of stable cycles and traces of transitory trajectories are readily discernible. Qualitatively, the relative intensities of cycles in each photograph are indicative of the time spent in each neighborhood. For these experiments, $\gamma=1$ and the value for the covariance of the white noise, σ, was determined from the rms output meter on the noise generator similar to the way described in [2].

Finally, the transitional process was recorded digitally in Fig. 10. A sample-and-hold network introduces a dc offset approximately proportional to the rms value of the diffusion process, in order to ensure nonzero inputs into the analog-to-digital con-

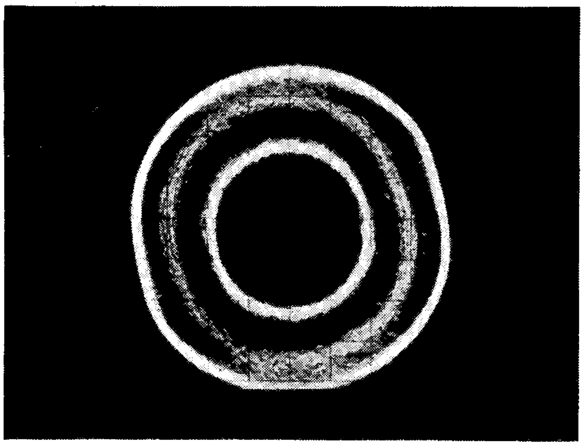

Fig. 9. Stochastic transitions in the phase plane as measured on an oscilloscope for the circuit of Fig. 6, $\gamma=2$. Scale setting was 0.5 V/div for $N=5$ with $a_1^2=3.3$ V, $a_2^2=2.33$ V, $a_3^2=1.93$ V, $a_4^2=1.61$ V, $a_5^2=0.403$ V, respectively.

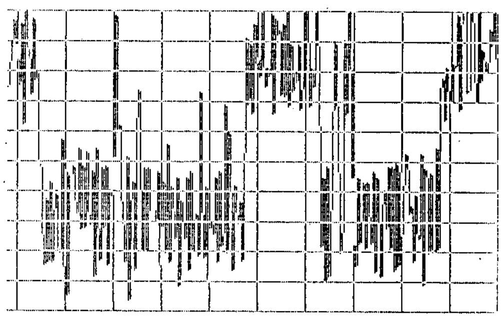

Fig. 10. Digitally recorded stochastic transitions for circuit of Fig. 6 and a_k's and γ as in Fig. 7. Time scale is 220 ms/div (horizontal axis). Vertical scale is arbitrarily adjusted for a convenient display.

verter. Transitions from one state to another are marked by step increases in the average value of the recorded signal due to the dc offset.

V. Conclusions

The possibility of experimentally investigating noise-perturbed oscillators with a given number of limit cycles was demonstrated. This work fills the gap between theoretically conccived physical situations and their realization. The authors believe that the above-described oscillators will allow for more principal experiments with white noise in coefficients a_k to observe the so-called stochastic postponements [2].

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Excess Phase Jitter Cancellation Method for SC Relaxation Oscillators

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***Abstract* — A practical jitter cancellation method for relaxation SC oscillators is presented. Two versatile switched-capacitor (SC) oscillators using the proposed excess phase jitter suppression technique are described. The jitter and the dependence of the oscillation frequency on the saturation voltages of operational amplifiers (op-amp) or comparators are eliminated. Therefore, there is no oscillation frequency limitation, except the one determined by the Nyquist rate. Because the frequency of oscillation of the proposed SC oscillators depends on a capacitor ratio and a reference voltage, the oscillators have excellent stability and accuracy. Experimental results showed good agreement with theoretical ones.**

I. Introduction

One practical technique for realizing precision monolithic circuits that has been recognized is switched-capacitor (SC) circuits using MOS technology [1]–[11]. Several SC nonfiltering applications have recently been presented [4]–[11]. These nonfiltering

Manuscript received September 2, 1986; revised February 11, 1987.

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IEEE Log Number 8714325.

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