On Transitions Between Quasilinear and Relaxational Regimes
in
Self-oscillators

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ABSTRACT - The Lienard (Rayleigh) equation for a self-oscillator with a number of limit cycles at small values of the dissipative parameter, $\mu > 0$, is considered for all values of $\mu \varepsilon (0,\infty)$. It is shown that essential qualitative changes in the structure of the phase plane can occur at intermediate values of μ, "on the way up" from quasilinear to relaxational regimes of oscillations. The illustration of such events along with classification of topologically conjugate equivalence classes of dynamical systems, and some numerical examples, are given.

1. **Introduction**. There are certain peculiarities regarding the operation of devices dependent on components with complicated nonlinear (current-voltage) characteristics; the performance of these devices "often ended in failure due to jumps..... oscillations and other exotic phenomena" [1]. On the other hand, modern solid state concepts [6], nonlinear circuitry, and certain areas of biochemical kinetics [3] give rise to the instruments and models [8] in which more than one stationary or periodical regime is possible. Obviously, the existence of multiple stationary states in electrophysical problems described in [8, 6 and references] in terms of spatial variables can lead to the above mentioned types of characteristics in the current-voltage domain. The purpose of this paper is to illustrate some concrete details of the relevant dynamics in the simple case of a self-oscillator governed by the Lienard equation:

$$\ddot{u} + \mu f(u) \dot{u} + u = 0 . \qquad (1)$$

Although the value of μ affects the nonlinear term, in the applied context (and accordingly, in the equations for dimensional variables), only parameters pertinent to linear components of the oscillator are varied, while the nonlinearity itself is invariable.

When the self-oscillator of (1) is a circuit, an invariant which reflects the physical properties of the nonlinear element is the "characteristic"

$$H(u) = \int_0^u f(s)\, ds . \qquad (2)$$

Details on the synthesis of pregiven characteristics and the role they play in oscillators can be found in [2], and a companion paper [13]. When the above mentioned parameters vary over a broad range (which is often unavoidable), reducing the problem to "small" or "large" values of μ can lead to inadequate conclusions and practical mistakes. There are several works on multiple limit cycles for the Lienard equation (see [9], [11]) however, none of them addresses the problem with parameter, μ.

The novelty of the information delivered by this paper consists in bridging the gap between two fundamental methods used in nonlinear circuit theory and practice. These methods, which are asymptotic by their nature, inevitably demand a "small" nonlinearity (quasilinear theory of nonlinear oscillations), or else, nonlinear terms dominate the equations (theory of relaxation oscillations).

From a qualitative point of view, no significant phenomena take place when μ in the equation (1) runs over $(0,\infty)$, provided that $f(\bullet)$ satisfies the conditions of the Levinson-Smith Theorem [12]. The "only" thing that happens is a strong "contortion" of a single limit cycle and an elongation of its period for large μ. Hence, under the above condition on $f(\bullet)$, Equation (1) generates dynamical systems which are topologically conjugate for all values of $\mu \varepsilon (0,\infty)$. This latter fact, is not in general valid for a broader class of nonlinearities

After standard transformations, (1) can be written in an equivalent form for integral curves:

$$\frac{dy}{dx} = \frac{-x}{\mu^2[y - H(x)]} \qquad (3)$$

where H is defined in (2).

All the limit cycles of (1), at "sufficiently small" values of $\mu > 0$, exist in the neighborhoods of bifurcating circles, whose radii are given by

$$\Pi(R) := \int_0^{2\pi} f[R \cos t] \sin^2 t \, dt = 0 \qquad (4)$$

On the other hand, all limit cycles of (1), existing for "sufficiently large" values of $\mu > 0$, have their limiting locations (on the phase plane for equation (3)) with $\mu \to \infty$ given by discontinuous solutions of the degenerate system:

$$y - H(x) = 0 \qquad (5)$$
$$\frac{dy}{dt} = -x \, .$$

As is well known, phase trajectories corresponding to discontinuous periodic solutions ("relaxational curves") of (5) coincide with certain arcs on the graph of the function H and their simple geometric construction does not involve solving of any differential equations [10].

A natural question is, "Do the limit cycles in the neighborhoods of the curves associated with (4) and (5) represent integral curves of topologically conjugate dynamical systems?" If the answer is negative, the next questions to ask would be, "How large, is the set, V, of different equivalence classes of dynamical systems generated by (1); what are the limit cycles of each of them; and, what is the mapping:

$\mu \, \varepsilon \, (0, \infty) \to V$?"

The answers to these questions can be induced immediately from the basic facts of structural stability theory for equations with special H.

A circuit-wise interpretation of these phenomena is partially consistent with the concept of "jump phenomena," with a major difference. In particular, the jumps from one equilibrium to another are completely controllable and predictable.

Hereafter, the function H is assumed to be odd ($H(x) = -H(-x)$), continuously differentiable, $H(x) \to +\infty$ when $x \to \infty$, and such that any of its positive roots is of odd multiplicity. It is also assumed that the number of roots is finite and if $r_k \, \varepsilon \, S$ ($k = 0,1,2,...,n$) with $r_0 = 0 < r_1 < r_2 <....< r_n$ where $S = \{ r_k \}$ is the set of all positive roots, then $dH(x)/dx$ has only one root on the interval $I_k = (r_k , r_{k+1})$ ($k = 0,1,2,...,n-1$). Equation (1) with such a function H determines a dissipative dynamical system for each $\mu > 0$ [4, Ch. 1, sec. 4.].

2. Conjugacy. A heuristic guess about plausible equivalence classes of topologically conjugate dynamical systems is helpful. Such classes are traditionally exemplified with explicitly defined limit cycles, in order to observe the qualitative evolution of phase portraits due to variations in the parameter's value. Bearing in mind the criteria for topological equivalence of dynamical systems [5], consider the following equation:

$$\ddot{x} + \dot{x}[(x^2 + \dot{x}^2 - a_2{}^2)^2 + 1 - \mu] \bullet$$
$$\bullet [x^2 + \dot{x}^2 - a_1{}^2 - (\mu + 1)^{1/2}] + x = 0 \qquad (6)$$

where $| a_1 | > | a_2 |$ and $\mu > 0$

The family of periodical solutions to (6):

$$x = (a_1{}^2 + (\mu + 1)^{1/2})^{1/2} \sin t$$

does not bifurcate, and for $\mu < 1$ there is only one limit cycle admitted by Equation (6). In the neighborhood of $\mu = 1$, two new periodic families

$$x = (a_2{}^2 + (\mu - 1)^{1/2})^{1/2} \sin t$$

and

$$x = (a_2{}^2 - (\mu - 1)^{1/2})^{1/2} \sin t$$

appear. Hence, for each $\mu > 1$, there are three hyperbolic limit cycles and therefore (6) is a member of the same equivalence class of dynamical systems, with parameter, as the system represented by Figure 1a and 1b below. Analogously, the equation

$$\ddot{x} + \dot{x}[(x^2 + \dot{x}^2 - a^2)^2 + 1 - \mu] + x = 0 \qquad (7)$$

is conjugate to the other dynamical system represented by Figure 1e and 1f. Resorting to such images noticeably reduces the time and efforts needed to plan and interpret computational work (at least it was the case with the authors).

3. Examples. In Figure 1 special types of dynamical systems are demonstrated through their qualitative portraits without axes scales and labels. Functions H_j are shown only in graphical form and exclusively to emphasize the essential qualitative differences between them. Below will be given explicit realizations of H_j providing phase portraits for "small" and "large" μ as they are pictured on Figure 1.

Functions H_j are determined as the following:

$$H_j(X) = \sum_k A_{jk} \exp((X - M_{jk})^2 / \sigma^2_{jk}) + X^3$$

for j = (1, 2)

$$H_j(X) = \sum_k{}' C_{jk} X^k$$ 'denotes sum over odd integers

for (j = 3, 4)

k =	7	5	3	1
j = 4				
C_{jk}	0.582052	-2.429340	2.914700	-0.985997

Table 1

Note: In Figure 1, when j = 1, then k = 4; when j = 2, then k = 6; when j = 3,4, then k = 7. For concreteness, a numerical simulation of the phase portrait for (3) with H_4 is provided in Figure 2.

4. Concluding remarks. The characteristics considered in the paper are by no means exotic objects with special properties. In many respects they have been met before in experimental work of the present and other authors [2, 7]. In the above considered examples, self-oscillatory regimes compose one or two conjugate classes.

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Figure 1. The qualitative types of phase portraits. All equilibria and limit cycles are of the hyperbolic type. The symbols UE, SE, SLC, and ULC indicate unstable equilibrium, stable equilibrium, stable limit cycle, and unstable limit cycle, respectively. The cases a, c, e, and g correspond to "small" μ in (3) for H_1, H_2, H_4, and H_3, respectively. The cases b, d, f, and h correspond to "large" μ in (3) for H_2, H_4, H_1 and H_3 respectively.

Figure 2. Numerical phase portraits, dx/dt vs x, for (3) with the characteristic H_4 of Table 1. a) at μ = 0.01, b) at bifurcational value of μ = 0.422 c) at μ = 2.0, d) characteristic $H_4(x)$ vs x with indicated relaxational curves.

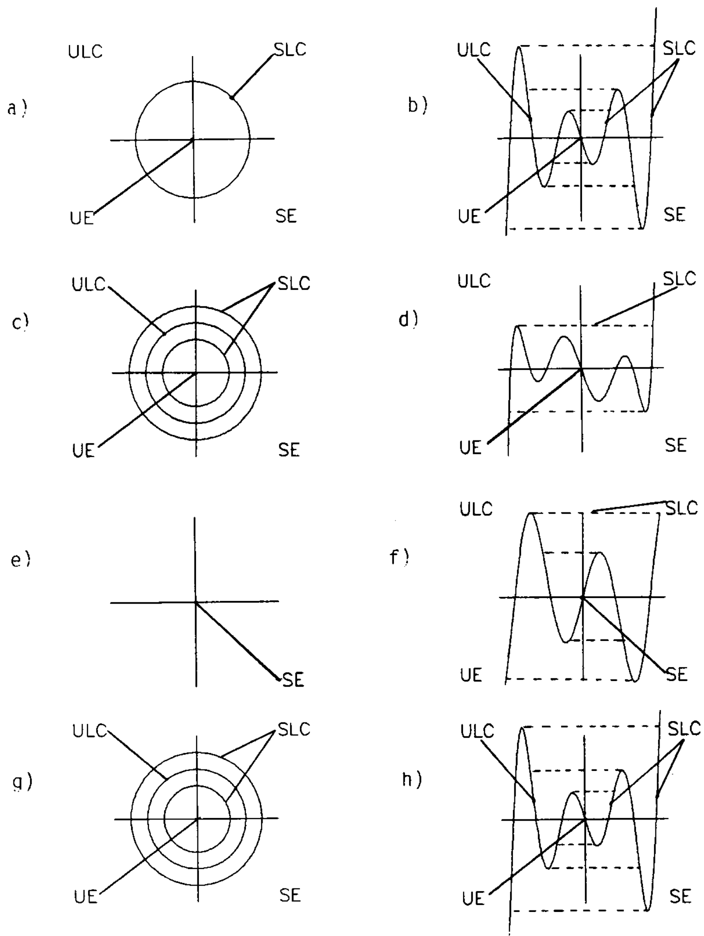

Figure 1

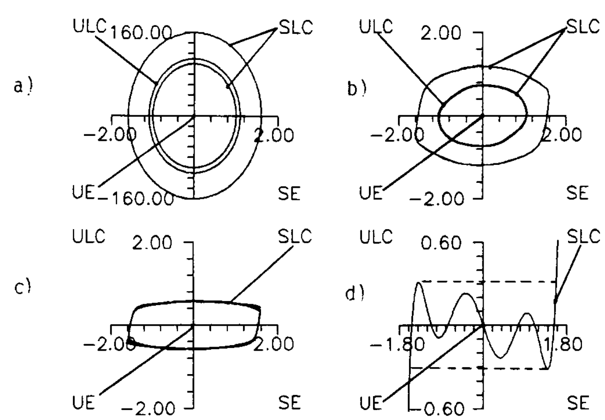

Figure 2