Multi-stable Periodical Devices with Variations on the Theme of Van der Pol

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ABSTRACT: *Two classes of nonlinear systems with one degree of freedom, each having several stable limit cycles, are introduced and analyzed. The realization of one of them is presented in the form of a simple compact electronic device. The "Van der Pol structured" equation having a given number of limit cycles with prescribed amplitudes is analytically synthesized in a framework of quasilinear theory. The qualitative structure of the phase space of such systems has dynamical properties similar to certain situations in chemical kinetics, applied physics and biology.*

I. Introduction

This paper focuses on the synthesis of several stable limit cycles in the phase space of a single device with one degree of freedom. The first chosen approach relies on a certain "distortion" of a familiar system, allowing one to obtain periodical regimes of a prescribed number and amplitudes. A similar approach was developed and applied to the analysis of randomly perturbed systems (**1**, **2**), allowing the authors to acquire in explicit analytical form a limit cycle as well as a solution of the corresponding Fokker–Plank Equation, in many cases of practical interest. One encounters findings closely connected to those of (**1**, **2**), in (**3**, **4**), under the name of "Improved Van der Pol Equation". The authors of these works investigate the synthesis of one limit cycle and do not refer to the earlier sources (**1**, **2**). As a second approach to the present paper, the special class of the Lienard Equations allowing an arbitrary number of cycles is introduced and a complete analytical synthesis for the case of polynomial nonlinearities is performed.

Systems with similar qualitative dynamics were recently discussed in different areas of applied sciences (**6**, **7**, **8**). However, one cannot point out any known concrete device of a simple physical nature suitable for reproducible and easily accessible experiments. The proposed circuit is convenient for introducing external perturbations and the subsequent observations of the output. Some numerical experiments are also presented to illustrate the different possibilities and the characteristic pecularities of both classes mentioned.

II. Analytical Models

Using the notations:

$$V_i(x;y) = \int_0^x g(s)\,ds + y^2 - E_i$$

where

$$E_i > E_{i+1};\, E_i > 0\ (i = 1, 2, 3, \ldots, n);\, g(x)x > 0;\, g(0) = 0,$$

consider the differential equation:

$$\ddot{x}+\mu\left[\prod_{i=1}^{n} V_i\left(x;\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\frac{\mathrm{d}x}{\mathrm{d}t}+g(x) = 0 \text{ with constant } \mu > 0. \tag{1}$$

Equation (1) or the corresponding equivalent system (2), in what will follow below, is termed the Conservative Multi-cycle Equation (CME), i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y;\frac{\mathrm{d}y}{\mathrm{d}t} = -w^2x-\mu\left[\prod_{i=1}^{n} V_i(x;y)\right]y. \tag{2}$$

Employing

$$\Phi(x, y) \equiv \int_0^x g(s)\,\mathrm{d}s+y^2$$

as a Liapunov function for the CME, one concludes:

(A) The solution ($x = 0$, $y = 0$) is unstable (stable) for odd (even) n, respectively.
(B) With odd (even) n, the limit cycles

$$\int_0^x g(s)\,\mathrm{d}s+y^2 = E_k$$

with odd k are stable (unstable) and unstable (stable) for even k, respectively.
(C) The domains of attraction of the stable limit cycles on the phase plane $x0y$, for odd (even) n, are separated from each other by the limit cycles with even (odd) numbers, respectively.

With the choice $g(x) \equiv w^2x$, the CME (1) has the set of solutions $x_k(t) = (\sqrt{E_k}/w)\sin(wt+\beta_k)$, $y = \sqrt{E_k}\cos(wt+\beta_k)$, with odd k that are orbitally asymptotically stable. The classical Van der Pol equation, with a single cycle, is received when $g(x) \equiv w^2x$, and $V(x;y)$ is replaced by $V(x;0) \equiv x^2-E_1$, with $n = 1$. Hence, it is quite natural to consider the "Van der Pol structured" Lienard Equation,

$$\ddot{x}+\mu D(x)\dot{x}+g(x) = 0 \tag{3}$$

where

$$D(x) = \prod_{i-1}^{n} V_i(x;0).$$

For the sake of abbreviation, Eq. (3) is termed the RME (Relaxational Multi-cycle Equation). In the absence of the analog to the Levinson–Smith theorem for existence of several stable limit cycles "in large", one can resort to the asymptotical theory with respect to the small parameter, $\mu \ll 1$. In the quasilinear case ($g(x) \equiv w^2x$), the bifurcation equation for the generating amplitude A of the periodical solution of the

RME is

$$\int_0^{2\pi} D[A\cos(u)]\sin^2(u)\,du = 0. \tag{4}$$

Since this is a polynomial equation, it does admit a number of different real roots under the proper choice of various values of parameters E_k. To obtain more concrete results, consider the Lienard Equation:

$$\ddot{x}+\mu f(x)\dot{x}+w^2x = 0, \tag{5}$$

with $f(x)$ belonging to the class of polynomial functions, and $\mu > 0$.

Lemma

Let $A_k > 0$, $A_{k+1} > A_k$ $(k = 1,\dots,N)$. There exists the polynomial,

$$f(x) \equiv a_{2N}x^{2N}+a_{2N-2}x^{2N-2}+\cdots+a_0,$$

in even degrees of x, with $a_{2N} > 0$ having N different simple positive roots b_k^2, such that for all sufficiently small μ, Eq. (5) allows N nontrivial periodical solutions, $x_k(t,\mu)$. These solutions have the following properties:

(1) $x_k(t+s_k, 0) = A_k\sin(wt+\Phi_k)$

where s_k are arbitrary constants and Φ_k depends on s_k.

(2) Every other periodical solution out of the sequence $\{x_k(t,\mu)\}$ including $x_N(t,\mu)$ is orbitally asymptotically stable. The coefficients a_{2k} of the polynomial $f(x)$ are given by Eq. (7) below.

Proof

Using the notations $R \equiv A^2$, and

$$K_{2k} \equiv \int_0^{2\pi} (\cos\Phi)^{2k}\sin^2\Phi\,d\Phi,$$

one obtains the bifurcational equation, with respect to R, as:

$$B(R) \equiv a_{2N}R^NK_{2N}+a_{2N-2}R^{N-1}K_{2N-2}+\cdots+a_0\pi = 0. \tag{6}$$

It is obvious, that in order for the polynomial, $B(R)$ in (6), with $a_{2N} > 0$, to have the roots A_k^2 $(k = 1,\dots,N)$, it is sufficient and necessary for the coefficients of the polynomial $f(x)$ to be:

$$a_{2k} = (-1)^K\frac{a_{2N}K_{2N}}{K_{2k}}\sigma_{N-k}(A_1^2, A_2^2,\dots,A_N^2) \tag{7}$$

$$(k = 0,1,\dots,N-1)$$

where σ_k are the values of the elementary symmetric polynomials calculated at the values A_k^2. Now, assume the coefficients a_{2k} are calculated by Eq. (7). Denote by

$$z_k \equiv \max_{0\leqslant t\leqslant T_k} |f_k(t)|,$$

where $f_k(t)$ is a periodical solution corresponding to the root A_k of bifurcational Eq. (6) and T_k is the period of the solution. Consider the system of intervals $[0, z_1]$, $[z_1, z_2], \ldots, [z_{N-1}, z_N]$. None of the limit cycles can be completely located within the dissipative or antidissipative domain, with respect to the Lebesque measure in the phase space of the RME (Bendixson Theorem). The question of dissipativity (or antidissipativity) is predetermined by the condition $f(x) > 0$ (or $f(x) < 0$), accordingly. Hence, each of the above intervals contains at least one root b_k, $z_{k-1} < b_k < z_k$. Since all the roots are simple,

$$f(x) = a_{2N} \prod_{k=1}^{N} (x^2 - b_k^2) \tag{8}$$

which completes the proof (9).

Note first that for the practical purposes of simulation, it is important to have the representation of $f(x)$, in form of (8), rather than,

$$f(x) = \sum_{k=0}^{N} a_{2k} x^{2k}.$$

In actual simulations, the polynomial $f(x)$ approximating some characteristics of real physical elements can, of course, be different from (8). However, as odd degrees of x do not effect the bifurcational equation, it is possible to ignore their presence in $f(x)$, regardless of whether one considers analysis or synthesis of the limit cycles. Hence, the class of polynomials, $f(x)$, in which the sum of even members creates a polynomial having all its roots positive, is the only one leading to the appearance of several stable cycles for small μ, in the Lienard Equation. Moreover, in synthesis, one can use only simple positive roots (with respect to x^2). So, the RME is indeed the direct and the natural generalization of the Van der Pol Equation. Since the substitution $x = (\mathrm{d}y/\mathrm{d}t)$ transforms the Lienard Equation to the Rayleigh Equation,

$$\ddot{y} + \mu F(\dot{y}) + w^2 y = 0,$$

where

$$F(y) = \int_0^y f(x)\,\mathrm{d}x,$$

all the facts from the Lemma can be reformulated and applied again, broadening the possibilities for simulation.

To compare the peculiarities of the RME and CME one should notice that the simplicity of the solutions of the CME do not depend on the value of μ. In the case of linear $g(x)$, all the periodical regimes have the same period. However, the periods are in general different for nonlinear $g(x)$. This also can (or should) be taken into account for the circuit design. On the other hand, the RME has a simple structure and the periods of its solutions depend on μ in any case, but in an extremely noticeable way for large μ. Simultaneously, with increasing μ, the limit cycle of the quasi conservative situation drastically deforms, as is very well known from numerical and asymptotical studies of the Van der Pol Equation (5).

III. Some Numerical Illustrations

To characterize certain internal parameters of the dynamical behavior of the systems, a limited numerical study was performed. The following data for the CME, with $n = 5$, reflect the time rates of convergence to the stable limit cycles. C_k, in Table I, denotes the cycle corresponding to the proper periodical solution of the CME with integral $E_k = k$ $(k = 1,\ldots,5)$.

TABLE I

			Γ_{in}			Γ_{out}	
μ	w	C_1	C_3	C_5	C_1	C_3	C_5
0.5	3	0.1590	0.4395	0.0340	0.2435	0.3220	0.0135
0.5	9	0.3235	0.5740	0.0350	0.3740	0.4195	0.0135
2.0	3	0.0375	0.0675	0.0085	0.0505	0.0600	0.0035
2.0	9	0.0381	0.0775	0.0085	0.0550	0.0670	0.0035

Γ_{in} and Γ_{out} describe the times of convergence within 1%, with respect to the value of $(V_k(x;y)+E_k)$. Initial conditions are chosen as $(\dot{x} = \sqrt{E_k}-0.5,\ x = 0)$ for calculation of Γ_{in} and $(\dot{x} = \sqrt{E_k}+0.5,\ x = 0)$ for calculation of Γ_{out}.

As seen from the data of Table I, for $\mu = 0.5$, the time of convergence from an unstable cycle to a stable cycle is less than one period of the cycle. The rate of convergence is significantly faster for large μ, for instance at $\mu = 2.0$, convergence results in less than 10^{-1} $(2\pi/w)$.

To evaluate the effect of a strong nonlinearity in the CME on the rate of convergence to the stable periodical regimes, the following procedure is employed. Assume that the time of transition (Γ_{in} and Γ_{out}) between two different positions in phase space is attributed to the linear oscillator with the time constant $2/h$. Then, h is determined from

$$(\sqrt{E_i}(\overset{+}{-})0.5)/(\sqrt{E_i}) = e^{-h}\Gamma_{out(in)}.$$

The ratios of the time constants for the linear oscillators, one with time constant $2/\mu$ and the other with time constant $2/h$, are presented in Table II.

It is known that a Van der Pol vibrator proceeds to a relaxational regime with increasing μ in Eq. (3). For $\mu > 1$, the cycle deforms significantly, and the period rapidly increases. Bearing in mind the presence of certain numerical obstructions when working with larger μ (5), the "embedding" of the RME is used, i.e.

$$\ddot{x}+\mu D_\sigma\left(x;\frac{dx}{dt}\right)\dot{x}+w^2x = 0 \tag{9}$$

where

$$\sigma\prod_{i=1}^{n} V_i(x;0)+(1-\sigma)\prod_{i=1}^{n} V_i\left(x;\frac{dx}{dt}\right) \equiv D_\sigma\left(x;\frac{dx}{dt}\right).$$

TABLE II

		$(2/\mu)$ / $(2/\lvert h_{in}\rvert)$			$(2/\mu)$ / $(2/\lvert h_{out}\rvert)$		
μ	w	C_1	C_3	C_5	C_1	C_3	C_5
0.5	3	8.719	0.830	6.198	3.330	0.957	14.120
0.5	9	4.285	0.635	6.021	2.168	0.735	14.120
2.0	3	36.968	5.402	24.791	16.058	5.138	54.463
2.0	9	36.386	4.705	24.791	14.744	4.602	54.463

At $\sigma = 1$, (9) converts to the RME and at $\sigma = 0$, (9) coincides with the CME. The use of embedding allows for more convenient numerical studies with $0 \leqslant \sigma \leqslant 1$ and arbitrary μ. Some of the results with $n = 3$, for the RME, are given below in Table III. Two stable periodical solutions are observed for only particular values of E_k. The dependence of the period on the value of μ is partly seen from the data below, with $E_1 = 0.1$, $E_2 = 1.0$ and $E_3 = 5.0$.

For the E_k chosen above, Fig. 1 demonstrates the evolution of limit cycles with growth of σ, as $\sigma \to 1$. Figure 2 indicates the stable cycles C_1 and C_3. Inspection of the numerical data indicates irregularity of the reciprocal location of adjacent limit cycles in the phase space. Further, there are strong limitations on the choice of values E_k for existence of cycles, for non-small μ.

IV. Devices

The above described two classes of nonlinear systems (CME and RME) are both realizable as compact electronic devices. Since realization of the CME requires a more complex structure, with the specific and novel properties of non-small nonlinearities, only the synthesis of the CME is presented here. The block diagram and the circuit drawing which describe the realization of the CME are shown in Fig. 3. This circuit was developed for $n = 5$, which yields three stable periodical cycles.

The circuit of Fig. 3 is composed of standard elements as shown. The multipliers

TABLE III

w	μ	Period
9	0.5	0.708
3	0.5	2.319
3	2.0	3.960

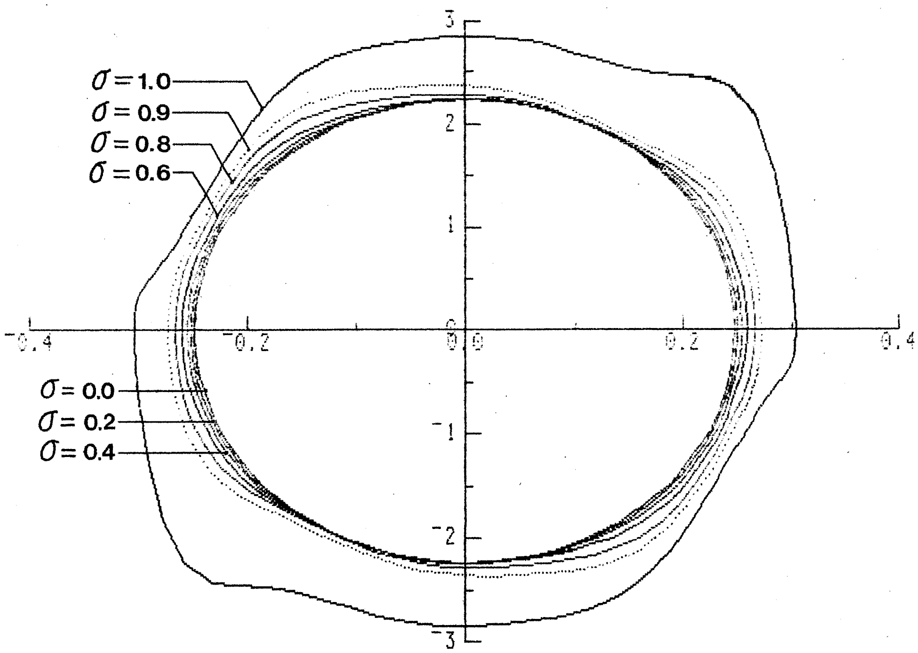


FIG. 1. Evolution of cycle C_3 with variations in σ for $w = 9$, $\mu = 0.5$.

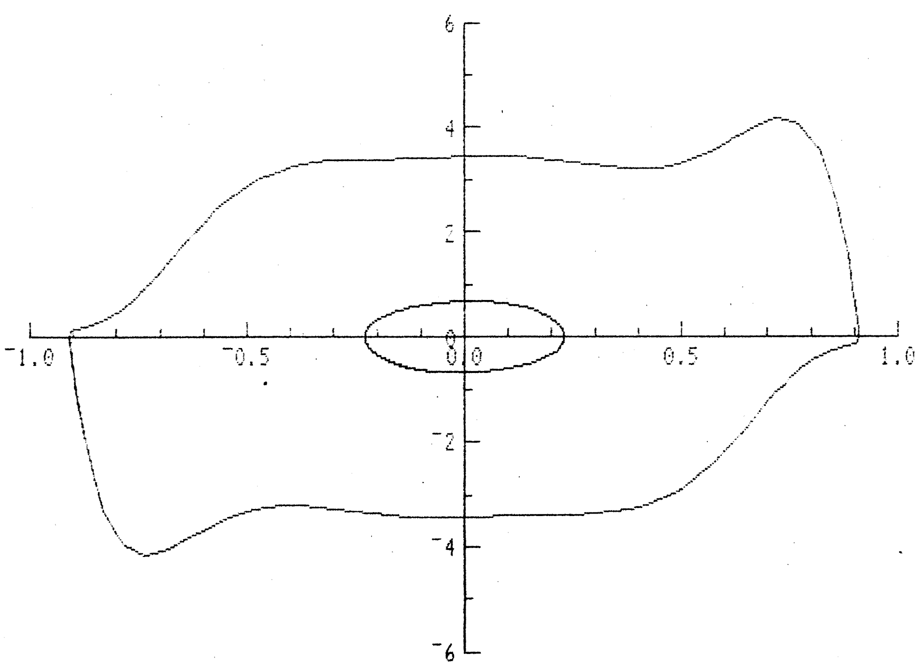
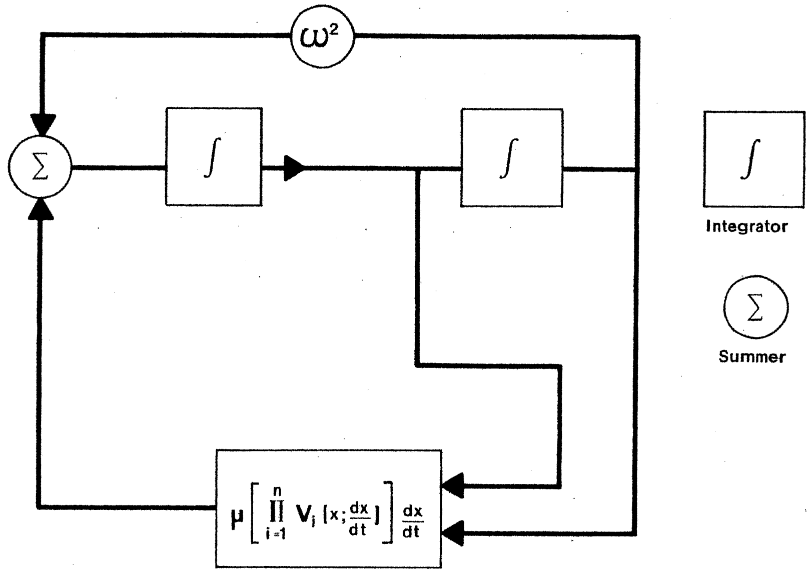
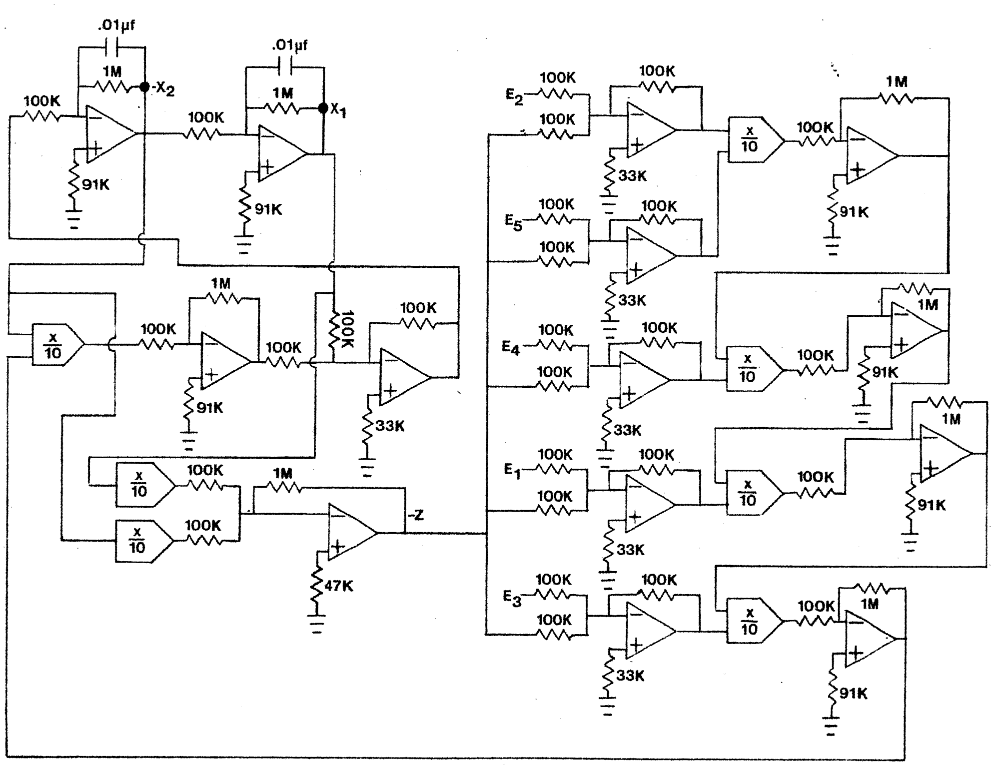


FIG. 2. Stable cycles C_1 and C_3 for $w = 3$, $\mu = 0.5$ in the RME.



(a)



(b) FIG. 3. Block and circuit diagrams for the realization of the CME.

and operational amplifiers each have an operational range of ± 10 V. Standard resistors ($\pm 5\%$) and capacitors were also utilized. The multipliers are of the transconductance integrated circuit type with a total error of $\pm 1\%$.

The main purpose of building the circuit is only to demonstrate the qualitative peculiarities of the internal dynamics of the CME. Therefore, in order to reduce the number of components necessary, w^2 and μ are chosen equal to 1. The time constant for the integrators is chosen as 0.001 s, which eliminates the need for further scaling in the circuit.

The choice of voltages, E_k, directly affects voltage levels elsewhere in the circuit. Hence, limitations exist regarding the range of values for the amplitudes of the limit cycles. In the course of performing a large number of experiments and measurements, the following characteristics were observed:

(1) The lower threshold for the amplitude of the limit cycle with the smallest amplitude is about 0.7 V. The upper threshold for amplitude of the cycle with the "largest" amplitude is 2.1 V. This determines the restrictions on E_k. Any combination of these five parameters within the above range lead to the predictable limit cycles.

(2) The voltage amplitude of the "smallest" cycle is subject to the greatest error with respect to the voltage amplitude predicted by the solutions to the CME. On the lower threshold, with the above indicated range for E_k, it is within 8%. As for the other cycles, the error is within 1% of the theoretical values.

(3) Measured frequencies of the cycles are in complete agreement with the chosen time constant parameter (time scaling parameter τ = RC).

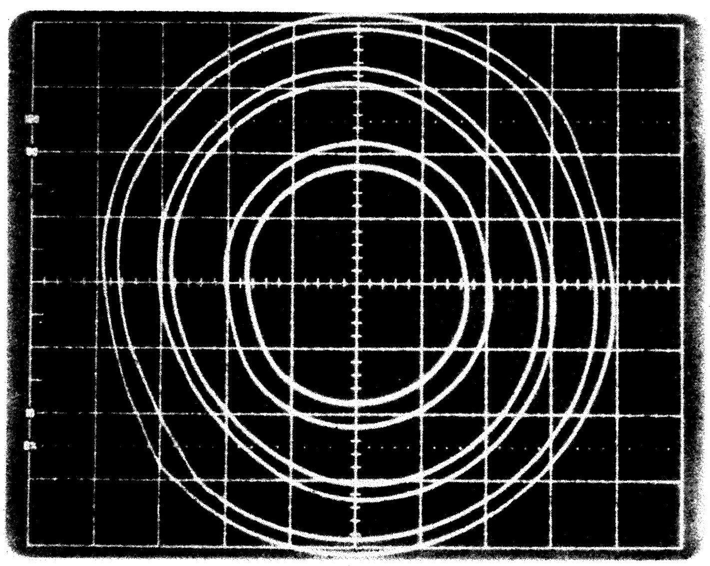


FIG. 4. Multiple exposure photograph for stable cycles in phase plane, for the CME, for two separate sets of coefficients E_k, for $n = 5$. Case 1 : $E_1 = 0.403$ V, $E_2 = 1.613$ V, $E_3 = 1.925$ V, $E_4 = 2.325$ V, $E_5 = 3.300$ V. Case 2 : $E_1 = 0.740$ V, $E_2 = 1.860$ V, $E_3 = 2.260$ V, $E_4 = 2.640$ V, $E_5 = 3.730$ V. Oscilloscope scale is 0.5 V/division.

Vol. 318, No. 6, pp. 373–382, December 1984
Printed in Great Britain

By changing the charge on the capacitor, at the point X_1 (Fig. 3), it is possible to vary the initial conditions. Therefore, controllable transitions from one stable cycle to another are readily performed. The time of convergence was not accurately measured. However, it was not possible to observe, on the oscilloscope at the working frequencies, any phase plane trajectories off the stable limit cycles.

For the sake of completeness of the description, two examples, representative of typical experiments performed on the circuit, are shown in Fig. 4. Here, stable cycles are superimposed in one photograph (multiple exposure) for easy comparison.

V. Conclusion

Recently, some authors have indicated the need to model the dynamics of systems met in certain applications (**6, 7, 8**) with several periodical regimes, particularly in the stochastic environment. To date, only very limited analytical means are available for revealing the peculiarities of such dynamics. The proposed circuits provide a simple means for direct experimentation on the interaction and the perturbed behavior of the above systems. Also, it may be possible to consider related devices as components in sensory and computational hardware or in adaptive control systems. For example, one can imagine a multi-stable circuit as a component for non-binary memory-like devices.

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