An Unorthodox Paradigm of a Relaxational Self-oscillator and Some Classes of Nonlinear One-ports

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ABSTRACT: *The design of components with a pregiven nonlinear current–voltage characteristic is presented. On this basis, hardware realization of self-oscillators capable of functioning in different periodical regimes, without change of internal parameters, is described along with analysis of these new types of oscillators. A new aspect of connections between quasilinear and relaxational oscillations is considered with concrete examples.*

I. Introduction

In Ref. (**1**) (p. 94) there is a discussion of devices with a certain nonlinear "input–output" relationship. One reads "... I want a nonlinear apparatus with that kind of characteristic. There are various ways of getting it, but the general principle involves using a push-pull." As we now know, no broad classes of nonlinearities have ever been synthesized by means of the technique suggested in (**1**). Ten years later, one finds in Ref. (**2**) an ample source of exercises on nonlinear circuits, often with formal nonlinear components of abstract, unclarified physical nature and rather complex characteristics. In a decade after, many new approaches were again discussed [see, for example (**3**)], aiming at the same purpose of constructing nonlinear devices with pregiven characteristics. Apparently, until recently, nonlinear components with the firm, requisite control over their characteristics did not exist as real hardware.

Throughout this paper, our interest is limited to the synthesis of nonlinearities for the purpose of constructing oscillators of simple topological structure and with a set of stable limit cycles. Oscillators of this type, working in a harmonical regime, have previously been described in (**4**) and proposed for some adaptive control applications in (**5**). In these devices, some limitations existed due to the use of multipliers having insufficient dynamic range. Presently, we describe oscillators that can work in a broad spectrum of different regimes, without multipliers.

There is an increasing interest for multiple steady-state electronic circuits. Some of the already existing developments will be specifically indicated below.

II. Synthesis of Nonlinearities

In linear circuits, avoidance of the inherent nonlinearities of semiconductor devices is achieved by limiting device operation to within the constitutive characteristics. For virtually any semiconductor device, however, if we do not choose the operating range appropriately, nonlinear regimes will be observed. Our purpose is to show how steadily reproducible characteristics can be used for receiving nonlinear one-ports with desirable and controllable properties.

We are interested in realizing nonlinearities with specific I–V characteristics in order to achieve predetermined operative functions. Since systematic and strictly justifiable procedures for synthesis of general nonlinearities do not exist, we shall expound the methodology used on a few concrete examples which will be of further interest in this paper.

Obviously, one is unlikely to find a single semiconductor one-port with an I–V characteristic as indicated in Fig. 1, and we develop a nonlinearity, like this, by combining multiple one-ports with "simpler" nonlinearities. The identifiable portions of the entire network, which have more simple I–V characteristics, are termed "sub-elements". Their topology and components are indicated in the schematics (see Figs. 2 and 4). The I–V characteristic of a sub-element is termed a "sub-characteristic".

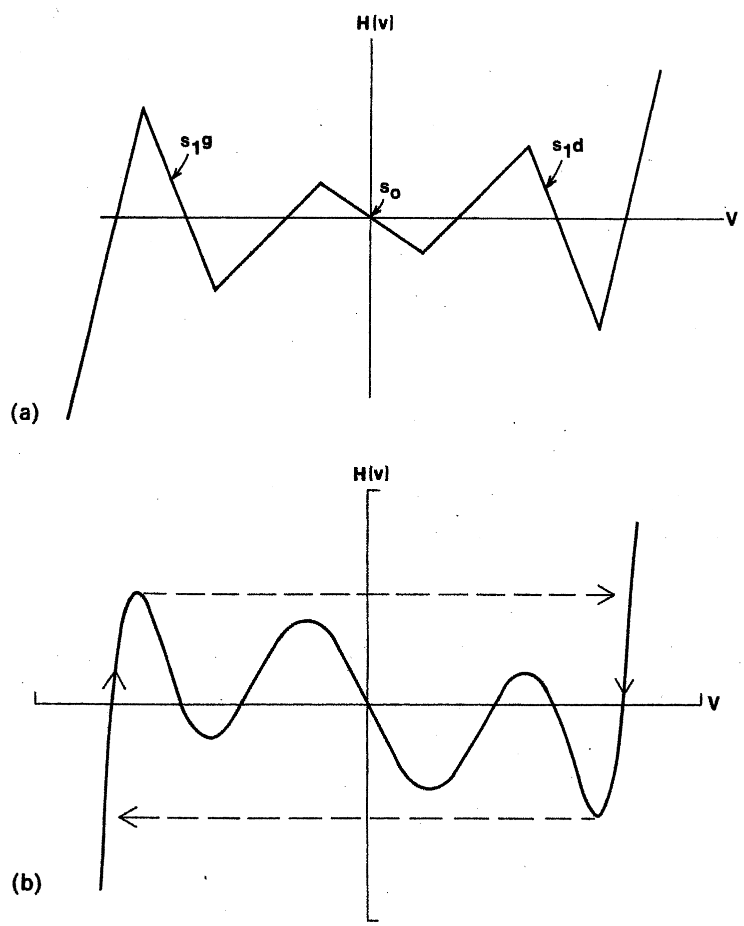

FIG. 1. (a) Piecewise linear I–V nonlinear characteristic; (b) nonlinear I–V characteristic with indicated relaxation oscillation.

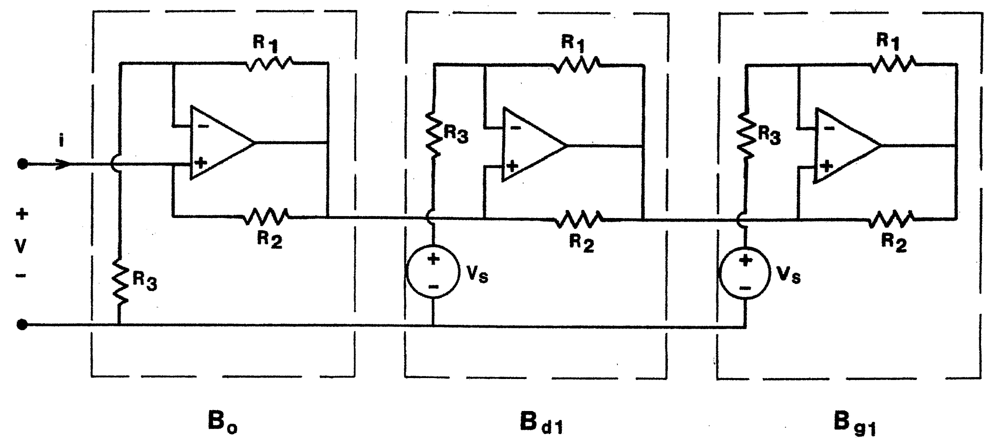

FIG. 2. Schematic for characteristic of Fig. 1a.

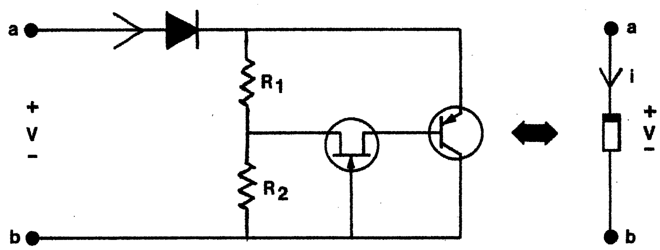

FIG. 3. Schematic for the nonlinear one-port component of Fig. 4.

In order to synthesize hardware with a more sophisticated nonlinearity, we divide the graph of the desired nonlinearity into several portions, and then choose a sub-characteristic (sub-element) to coincide with each portion. Choosing suitable sub-characteristics is the subject of extensive trials and numerical simulations. A sub-element is itself a one-port device, and with suitably chosen sub-characteristics, the parallel connection of all sub-element results in an approximation to the desired characteristic as a whole.

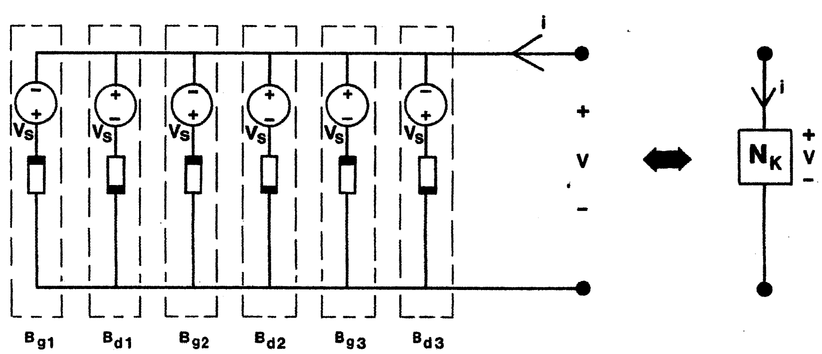

FIG. 4. Nonlinear one-port device used in circuit of Fig. 6 to produce desired multiple relaxation and quasiharmonic oscillations.

We briefly indicate the performance of the described stages in an attempt to synthesize the nonlinearity shown in Fig. 1a. To begin with, we choose the circuit of Fig. 2, which is a combination of similar sub-elements with different values of participating parameters. We now select a certain set of concrete values of these parameters as a "zero step" in a sequence of iterations (which follows). Note that the characteristic of this circuit, Fig. 2, depends on the values of the resistors of each sub-element. In their turn (if this characteristic is assumed to approximate the characteristic of Fig. 1a), the relationships between the values of the resistors are predetermined by the values of the negative slopes, s_k, of the particular portion of the desirable piecewise linear characteristic of Fig. 1a. Hence,

$$s_0 = -\frac{R_1}{R_2R_3} \quad (\text{for } B_0), \tag{1}$$

and the s_ks and v_ss are determined analogously for the sub-characteristic of the B_{dk} and B_{gk} sub-elements, respectively. The purpose of the subscripts g and d is to indicate which hardware sub-element corresponds to a particular portion of the characteristic with $g(d)$ denoting the part of the characteristic in the left (right) half plane. Now, the nonlinearity of Fig. 1a, or a similar one, can be achieved by an iterative procedure with regard to the values of the s_ks and the v_ss. Note that the voltage sources, v_{sk}, provide for translation of the I–V characteristic in the plane.

The characteristic realized and tested in Ref. (**6**, Fig. 11) (which was not received for a specific reason, but happens to serve as a useful example) is of interest as concrete evidence of practical possibilities for creating nonlinearity.

Having established, in principle, one method of synthesis, we will indicate alternative design procedures. In Refs. (**7–9**), numerous elementary negative resistance devices, which can be used as elementary "bricks" for building a broad class of one ports, are described.

We concentrate now on the construction of a broader class of nonlinearities, different from the piecewise linear class already described. As an example, consider the characteristic in Fig. 5. Choosing the sub-element of Fig. 3 with concrete voltage (polarity and magnitude), we develop the circuit of Fig. 4. After several approximations, the I–V characteristic of Fig. 5 is reached by means of the computer program SPICE, in the same manner as was described for piecewise linear synthesis. Unfortunately, there are difficulties in comparing the I–V characteristic determined by SPICE with the actual device characteristic, for a circuit of this complexity. This problem was previously noticed and partly alleviated in (**6**).

Remark on possible applications

In automatic control systems, the usual sources of nonlinearities are concentrated in the servomechanisms and sensory devices. So great, so significant was the impact of the lack of information regarding these technical components that it led to an excessively broad formulation of the problem of global stability and a dependence on exceedingly general classes of admissible nonlinearities [(**10**), Chap. 1]. Complete control over even a part of the nonlinearities in the system allows for a better

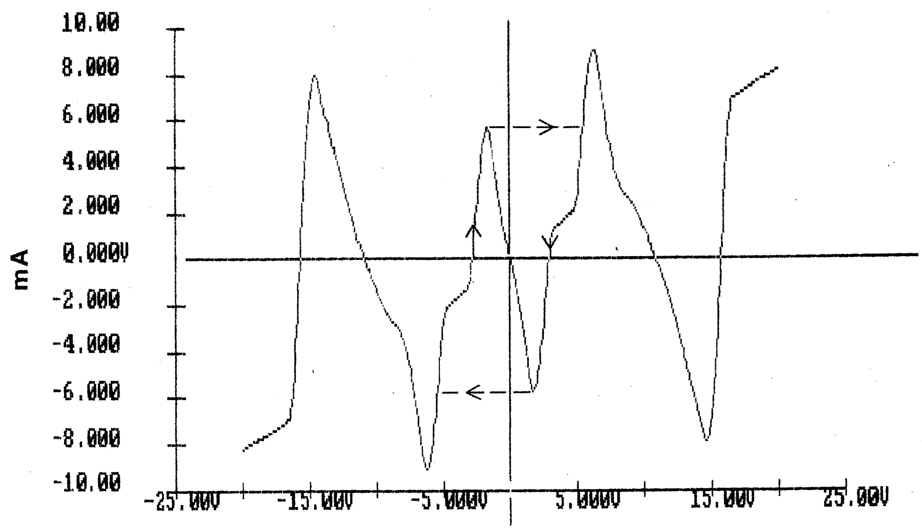

FIG. 5. I–V characteristic resulting from one-port of Fig. 4, determined with SPICE, for a particular choice of Rs and V_ss, in order to produce two quasiharmonic oscillations and one relaxational oscillation. Expected relaxation oscillation is indicated.

perception of dynamical peculiarities and a simplified design. One example of such an approach is given in (**5**).

III. On Self-oscillators and Applications

In Fig. 6, we show the principal circuit for the self-oscillator under discussion. If $H(v)$ is an I–V characteristic of a voltage controlled nonlinear one-port device, the governing equation for the circuit is:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} + \left(\frac{L}{C}\right)^{1/2} \frac{\mathrm{d}[H(v)]}{\mathrm{d}\tau} + v = 0, \tag{2}$$

where t (sec) $= w\tau$ and $w^2 = 1/(LC)$. In this model, we have included the internal resistive losses of the inductor and capacitor in the middle term of Eq. (2). For this Lienard type equation there exist two different constructive asymptotic theories allowing for the determination of all periodical solutions of (2). Recently, new and general results, unrelated to asymptotic methods, on the existence of N limit cycles of (2) were presented in (**11**). References (**11**) and (**12**) are the only sufficiently broad investigations known to the authors. With regard to our purpose, the following comments are appropriate. The results of the above works allow one to construct a nonlinearity with a number of limit cycles; however, since the methods and results of these works are predominantly of a qualitative nature, they provide little information about the character of the periodical solutions themselves.

For these reasons, we resort to asymptotic theories. In the Lienard equation with parameter μ:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}s^2} + \mu \frac{\mathrm{d}}{\mathrm{d}s} H(v) + v = 0 \tag{3}$$

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we assume $H(v)$ is an odd function. By means of the substitution $s = \mu\tau$ and

$$u(\tau) \equiv \int_0^\tau x(\xi)\,d\xi$$

one receives

$$\varepsilon\ddot{u} + H(\dot{u}) + u = 0 \tag{4}$$

where

$$\varepsilon \equiv \frac{1}{\mu^2}.$$

Letting $du/d\tau \equiv -y$, we come to the equivalent system:

$$\frac{du}{d\tau} = -y \qquad \varepsilon\frac{dy}{d\tau} = u - H(y). \tag{5}$$

We emphasize that in system (5), function H describes the same voltage-controlled nonlinear characteristic as in the oscillator of Fig. 6. Thus, having a given nonlinear characteristic the search for periodical solutions in the framework of asymptotic theory of relaxational oscillations can be done only on the basis of the characteristic itself, without resorting to the system of differential equations [(**13**), Chap. 3]. The construction of these periodical solutions is well known and mathematically justified (**13, 14**). Therefore, we only indicate a set of limit cycles on the phase plane with a given characteristic, see Figs. 5, 8 and 1b. The periods of these limit cycles are representable by an asymptotic expression, with respect to ε in (5), and are given in (**13**).

Utilizing the characteristic of Fig. 8 in the circuit of Fig. 6, one notices that the relaxation oscillator with such a nonlinearity should admit two stable limit cycles separated by an unstable one [a detail missing in the monograph (**15**), fig. 76] and an unstable equilibrium at the origin.

Remark

To justify the indicated construction of the limit cycles, choose a sufficiently large $\varepsilon > 0$ in (5), then due to (**13**) (Theorem 12), there will be two stable limit cycles of the system (5) lying in the neighborhood of the two indicated "relaxational limit cycles". Due to the Bendixson–Poincare theorem these limit cycles should be separated by another unstable limit cycle. Using the transition, $\varepsilon \to 0$, and

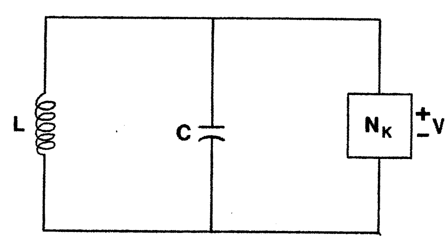


FIG. 6. Multi-state oscillator circuit.

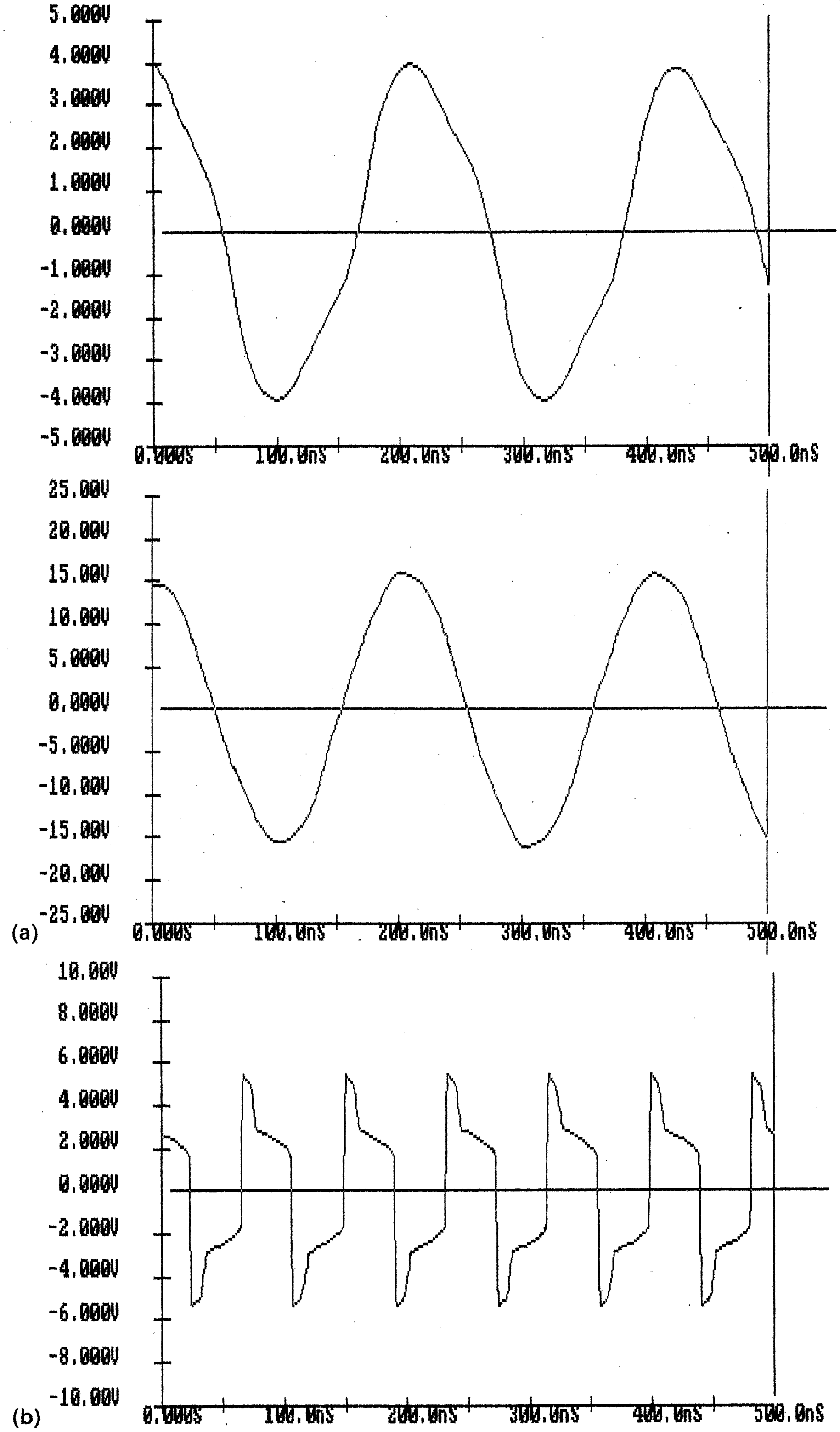


FIG. 7. SPICE simulation outputs for oscillator of Fig. 6 [Eq. (3)], with characteristic of Fig. 5: (a) quasiharmonic oscillations for large C (sufficiently small μ), (b) relaxation oscillation for small C (sufficiently large μ).

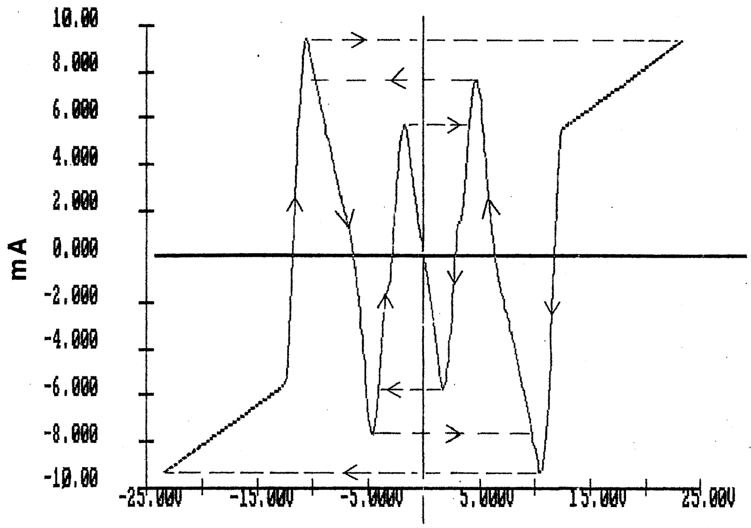


FIG. 8. *I–V* characteristic resulting from one-port of Fig. 4, determined with SPICE, for a particular choice of *R*s and V_ss, in order to produce two stable relaxational oscillations. Expected stable limit cycles, and the unstable cycle in between, are indicated.

considering the oscillator

$$\varepsilon\ddot{u} - H(\dot{u}) + u = 0 \quad (\text{instead of } \varepsilon\ddot{u} + H(\dot{u}) + u = 0)$$

for which the intermediate cycle is stable, allows one to find the limiting location of this limit cycle, as shown in Figs. 1b, 5 and 8.

The realization of the oscillators based on the described nonlinearities has not met with any significant difficulties, particularly in the case of one relaxational limit cycle. Therefore, the ensuing considerations are devoted to more refined phenomena, discovered in multiple cycle oscillators. Between two great asymptotic theories, quasilinear and relaxational, lies a "no-man's land" of intermediate values of μ in (3).

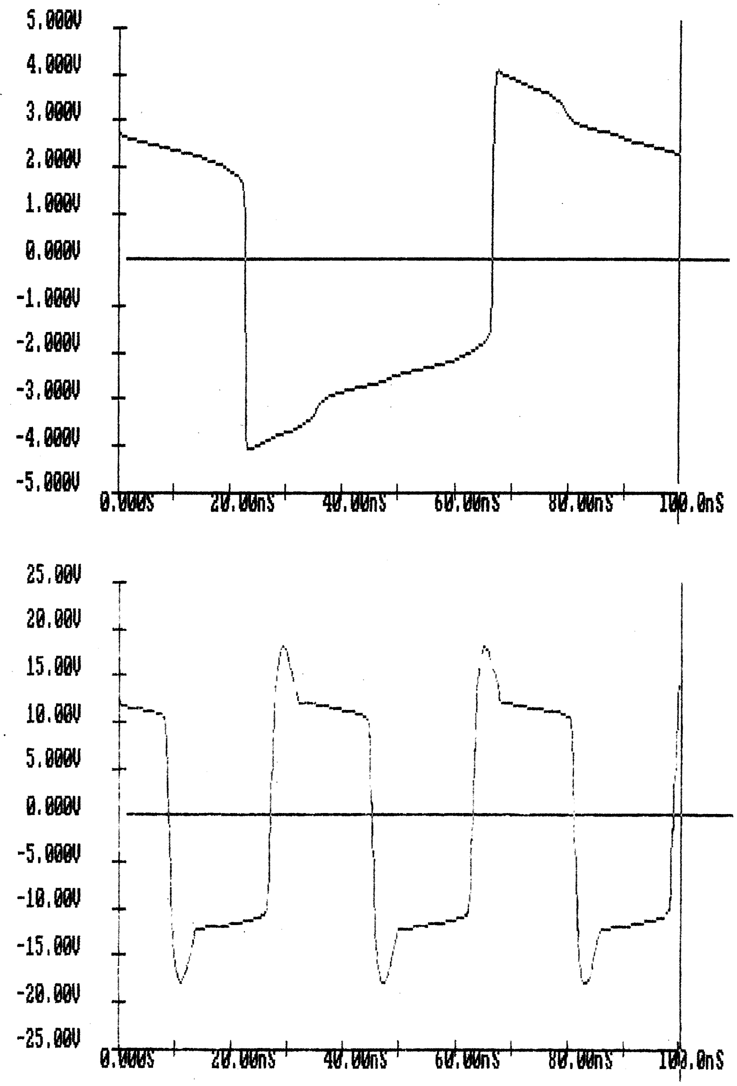
Consider the characteristic H such that the oscillator of (3) has two stable limit cycles of quasilinear nature [for sufficiently small μ in (3)]. Such a characteristic is indicated in Fig. 5. We are certain to have two stable quasiharmonic oscillations (with approximate amplitudes of 5 and 13 V), based on the theoretical considerations outlined in (**4**, **5**, **13** and **16**). Note that by increasing C, we decrease μ for the oscillator of (3). For sufficiently small μ, to insure operation in the quasi-harmonic regime, the results of a SPICE simulation reveal the two stable oscillations shown in Fig. 7a, for the characteristic of Fig. 5. As the value of C is decreased (μ increased), the oscillations become more "relaxational" as indicated in Fig. 7b. These results were confirmed experimentally with hardware.

For sufficiently large μ, according to the characteristic in question, oscillator (3) admits only one stable relaxational periodic solution [see (**3**), Theorem 12]. Hence, in proceeding from small μ in (3) to large values of μ, the oscillator undergoes a qualitative change in the structure of the phase plane. There exists the value of

$\mu = \mu^*$ when all three periodical solutions, which existed for smaller values of μ, are replaced by a stable single limit cycle.

The authors are not able to provide a detailed picture of the changes in the dynamical behavior and structure of the phase space for the oscillator prior to reaching the value of $\mu = \mu^*$, nor are they aware of the existence of such an investigation in the literature. The approximate estimate of μ^* however, can be done on the basis of the numerical technique developed by the authors and not discussed here.

As a final example, the characteristic of Fig. 5 is modified, as shown in Fig. 8. Clearly, two relaxational oscillations are expected for sufficiently large μ. The results of a SPICE simulation are shown in Fig. 9. The oscilloscope tracings of Fig.



(a) (b)

FIG. 9. SPICE simulation outputs for oscillator of Fig. 6 [Eq. (3)], with characteristic of Fig. 8: (a) small amplitude relaxation oscillation, (b) large amplitude relaxation oscillation.

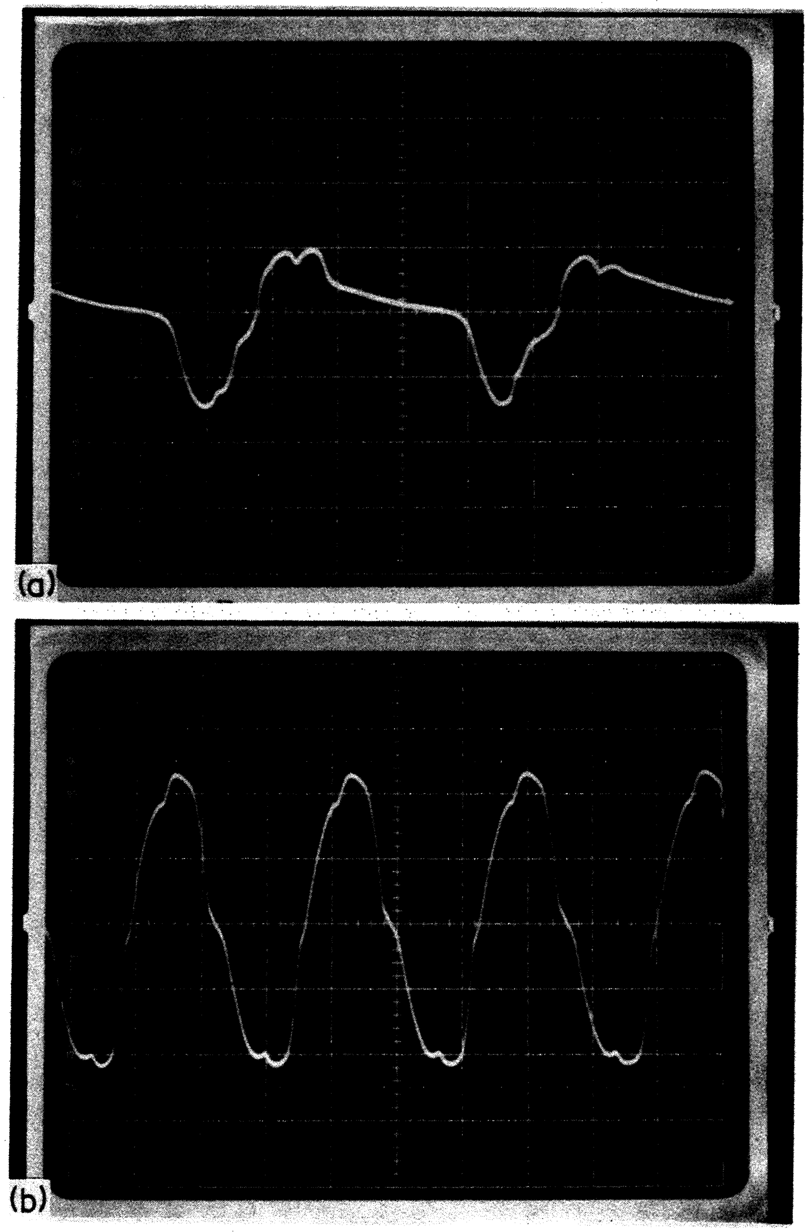


FIG. 10. Oscillator as described in Fig. 9. Oscilloscope settings are 5 V/division and 1 μs/division: (a) small amplitude relaxation oscillation; (b) large amplitude relaxation oscillation.

10 were produced by a hardware realization of the proposed circuit. Discrepancies are due primarily to our inability to control the device characteristics of the JFET. Such control, however, is completely within reach, at the manufacturing level. Qualitative agreement, however, is excellent!

It is interesting to note that in our experiments with these oscillators, a decrease in C [in (5) $\varepsilon = C/L$] causes an increase in the observed period, contrary to a situation with quasilinear oscillations and in complete agreement with relaxation theory [(**13**), Formula 8.7, Chap. 3]. The period, for sufficiently small ε, is (asymptotically) growing linearly in ε.

Remarks on motivations and applications

Physical investigations of phase transitions in macroscopic molecular or biological systems lead to dynamic models which are hardly tractable analytically. Essential components of these models are nonlinear oscillators with a set of stationary states, equilibria or periodical motions. Seemingly, the probable means of analysis of the above systems is a numerical computational approach. However, digital computers become over-burdened as required throughput reaches the level necessary for these numerical experiments. As a remedy, a current tendency is to mimic dynamical phenomena with nonlinear electronic devices. For a description of such a simulation, in noise induced phase transitions, we refer to (**17**). So far, similar work on auto-oscillatory systems has stagnated due to the absence of multi-limit cycle oscillators. For theoretical analysis of such models, we refer to (**18**). Another source of oscillators with multiple limit cycles orginates in the area of biochemical kinetics and models of self-organization in biology (**19**, **20**).

IV. Conclusion

This work has shown the possibility of the realization of complicated nonlinear characteristics. While there are no systematic algorithms for performing this task, certain patterns of characteristics were nevertheless built and verified in detail on oscillators having a predictable set of limit cycles. Some applications illustrating the theory have been given.

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