Int. J. Non-Linear Mechanics, Vol. 26, No. 5, pp. 701–709, 1991
Printed in Great Britain.

0020-7462/91 $3.00 + .00

QUASILINEAR AND RELAXATIONAL REALMS IN MULTIPLE REGIME SELF-OSCILLATORS

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(*Received* 27 *September* 1989; *in revised form* 22 *June* 1990)

Abstract—The results of this paper are new in that they suggest a dissipative non-linearity for which a given number of quasilinear limit cycles of the Lienard (Rayleigh) equation is accompanied by a given (and in general different) number of relaxation regimes. These aspects of global behavior of the Lienard equation with parameter have not been previously recorded in the literature on non-linear oscillations. Thus, qualitative dynamics, whose existence otherwise could be questioned or only hypothesized as possible "in principle" are delivered by concrete oscillators. Details of global bifurcations at intermediate values of the relaxational parameter along with conjugate classes and certain conjectures are presented on the basis of several numerical experiments for different qualitative situations.

1. MOTIVATIONS AND OVERVIEW

There are certain peculiarities regarding the operation of devices dependent on components with complicated non-linear (current-voltage) characteristics; the performance of these devices "often ended in failure due to jumps . . . . oscillations and other exotic phenomena" [1]. On the other hand, solid state concepts [2], non-linear circuitry [3], and certain areas of biochemical kinetics [4] give rise to the instruments and synthesized models of neural units [5] in which more than one stationary or periodical regime is possible. The purpose of this paper is to illustrate some concrete details of the relevant dynamics in the simple case of a self-oscillator governed by the Lienard equation:

$$\ddot{u} + \mu f(u)\dot{u} + u = 0. \tag{1}$$

Methods for determination of the number and locations of limit cycles in two asymptotic cases related to the parameter $\mu \to 0$ and $\mu \to \infty$ are well known. The value of μ in (1) affects the non-linear term, but in the applied context (and accordingly, in the equations for dimensional variables), only parameters pertinent to linear components of the oscillator are varied, while the non-linearity itself is invariable. For this former reason, the constraints on μ of being "small" or "large" are not always adequate. In a circuit-wise interpretation of (1), for instance, an invariant which reflects the physical properties of the non-linear element is the "characteristic"

$$H(u) = \int_0^u f(s)\,\mathrm{d}s. \tag{2}$$

Details on the synthesis of characteristics and the role they play in oscillators can be found in [6], which is a precursor of the present investigation. There are several works on multiple limit cycles for the Lienard equation (see [7], [8]) however, none of them addresses the problem with parameter μ.

The outline of this paper is as follows:

(a) Section 2 presents the class of oscillators which are topologically equivalent [9] to the Lienard oscillators under investigation. The chosen representatives of equivalence classes allow explicit periodical solutions in closed form for all values of the parameter μ. Therefore, according to Peixoto theorem [10], they demonstrate the qualitative nature of the results which are given later. The rest of the paper is devoted to confirming the above equivalence.

(b) In Section 3 we describe how to construct non-linear characteristics that produce a spectrum of concrete phase plane pictures, which constitute the main purpose of this

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Contributed by A. H. Nayfeh.

paper and the novelty of its results. These pictures confirm the suggested equivalence mentioned in item (a), above. Also in this section, we give the formulation and proofs of a few statements which serve as a useful tool in the organization of the numerical experiments. They are not necessarily closely related.

(c) Section 4 numerically verifies the central points of our paper. The tools of item (b) above were helpful for the control of numerical solutions and bifurcation values of the parameter μ. For better understanding of the figures in the results, we define the following notation:

$kQsR$: means that an oscillator has "k" quasilinear limit cycles and "s" relaxational periodical regimes, where "k" and "s" are non-negative integers.

2. BASIC RESULTS

From a qualitative point of view, no significant phenomena take place when μ in the equation (1) runs over $(0, \infty)$, provided that $f(\cdot)$ satisfies the conditions of the Levinson–Smith theorem. The "only" thing that happens is a strong "contortion" of a single limit cycle and an elongation of its period for large μ. Hence, under the above condition on $f(\cdot)$, equation (1) generates dynamical systems which are topologically conjugate for all values of $\mu \in (0, \infty)$. This latter fact is not in general valid for a broader class of non-linearities. The results of this paper are new in that they suggest a dissipative non-linearity for which a given number of quasilinear limit cycles is accompanied by a different (yet predetermined) number of relaxation regimes. Examples of such behavior were previously unknown.

After the standard transformation:

$$t = \mu\tau$$

$$-y = \frac{\mathrm{d}x}{\mathrm{d}\tau}$$

$$x(\tau) = \int_0^\tau u(\zeta)\mathrm{d}\zeta$$

(for details see [6], equations (4) and (5)), (1) can be written in an equivalent form for integral curves:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{\mu^2[y - H(x)]} \tag{3}$$

where H is defined in (2).

All the limit cycles of (1), at "sufficiently small" values of $\mu > 0$, exist in the neighborhoods of the circles [11 and 12, Section 3.3.2], whose radii are given by

$$\prod(R): \; = \int_0^{2\pi} f[R\cos t]\sin^2 t\,\mathrm{d}t = 0. \tag{4}$$

Such circles are called bifurcational circles [11, 13].

On the other hand, all limit cycles of (1), existing for "sufficiently large" values of $\mu > 0$, have their limiting locations (on the phase plane for equation (3)) with $\mu \to \infty$ given by discontinuous solutions of the degenerate system:

$$y - H(x) = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x. \tag{5}$$

As is well known, phase trajectories corresponding to discontinuous periodic solutions ("relaxational curves") of (5) coincide with certain arcs on the graph of the function H and their simple geometric construction does not involve solving of any differential equations [14].

The Peixoto theorem on conjugacy is helpful in planning and interpretation of computational work by allowing one to resort at first to hypothetical models and intuitive guesses.

As will be shown below, the following systems proved to be conjugate to the Lienard oscillators considered in the paper. Consider

$$\ddot{x} + [(r^2 - 2)^2 - (A(\mu) - 0.5)][r^2 - 1 - \mu]\dot{x} + x = 0 \tag{6}$$

where

$$r^2 = \dot{x}^2 + x^2$$

and

$$A(\mu) = \frac{2}{\pi}\tan^{-1}\mu, \quad \mu > 0.$$

The family of periodical solutions to (6),

$$x = (1 + \mu)^{1/2}\sin t,$$

does not bifurcate and for μ with $A(\mu) < \frac{1}{2}$, this is the only limit cycle admitted by (9). In the neighborhood of $\mu = \mu^*$, $(\mu^*: |A(\mu^*) = \frac{1}{2})$, two new periodic families

$$x = (2 + / - \sqrt{A(\mu) - \tfrac{1}{2}})^{1/2}\sin t$$

appear. (Equivalently, for this example, $\mu^* = 1$.) Thus for each $\mu > \mu^*$, there are three hyperbolic limit cycles, a situation presented on Fig. 1.

Analogously, the equation

$$\ddot{x} + [(r^2 - a^2)^2 + 1 - \mu]\dot{x} + x = 0 \tag{7}$$

is conjugate to the Lienard oscillator with the phase portrait of Fig. 2. Similar considerations are valid for other cases. On the basis of our experiments, we propose the following conjecture.

> Under certain conditions on H (which follow next), the number of limit cycles of the Lienard equation is a monotone function of μ and all bifurcations are of the fold type.

As will be seen, all our numerical experiments are in agreement with this conjecture.

Since the Van der Pol equation presents the simplest oscillator with one limit cycle, and for this case the characteristic H is odd, we conserve this property of H for the case of

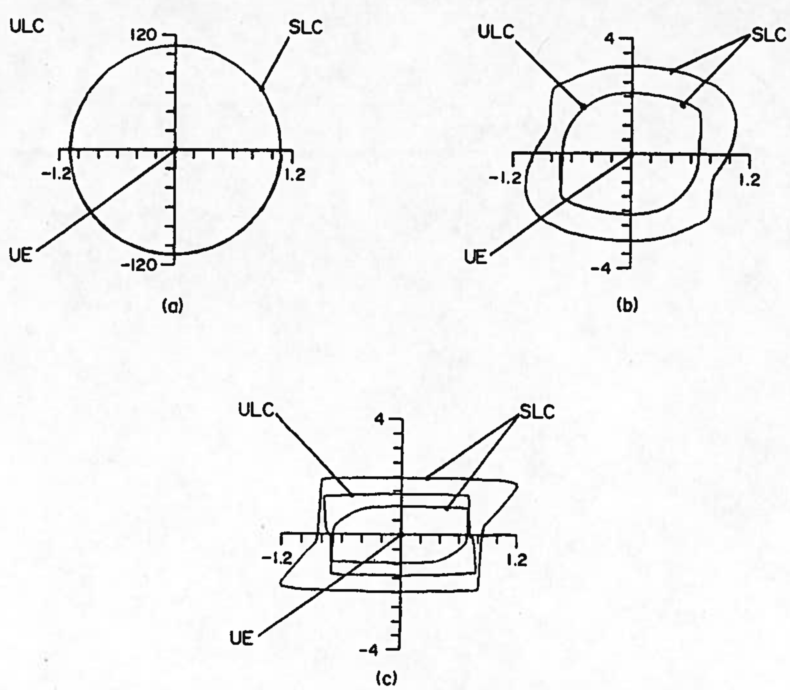

Fig. 1. Numerical phase portraits, dx/dt vs x, for $1Q3R$ oscillator with the characteristic H_2 of Table 1 (a) at $\mu = 0.3$, (b) at bifurcational value of $\mu = 2.2$, (c) at $\mu = 5$.

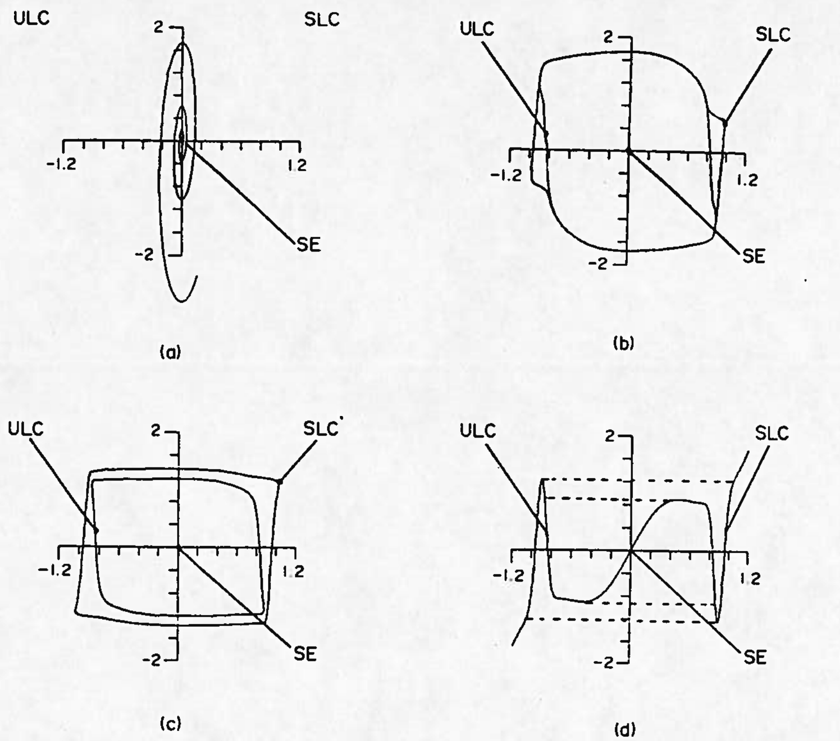

Fig. 2. Numerical phase portraits, dx/dt vs x, for $0Q2R$ oscillator with the characteristic H_1 of Table 1 (a) at $\mu = 0.01$, (b) at bifurcational value of $\mu = 1.4832$, (c) at $\mu = 4.0$, (d) characteristic $H_1(x)$ vs x with indicated relaxational curves.

multiple limit cycle oscillators. Hereafter, the function H is assumed to be odd ($H(x) = -H(-x)$), continuously differentiable, $H(x) \to +\infty$ when $x \to \infty$, and such that any of its positive roots is of odd multiplicity. It is also assumed that the number of roots is finite and if $r_k \in S$ ($k = 0, 1, 2, \ldots, n$) with $r_0 = 0 < r_1 < r_2 < \ldots < r_n$ where $S = \{r_k\}$ is the set of all positive roots of $H(x) = 0$, then $dH(x)/dx$ has only one root on the interval $I_k = (r_k, r_{k+1})$ ($k = 0, 1, 2, \ldots, n-1$). Equation (1) with such a function H determines a dissipative dynamical system for each $\mu > 0$ [15, Ch. 1, Section 4].

3. TECHNICAL BACKGROUND FOR THE CASE WITH A CHARACTERISTIC HAVING 3 POSITIVE ROOTS

In this section, H with three different positive roots is considered. Let, $0 < Q_1 < Q_2 < Q_3$ be notations for maximum values of $|H(x)|$ on $[0, \infty)$ with $Q_k = |H(m_k)|$, $k = 1, 2, 3$. The rest of the notations preserve their meaning from the above. There are six different classes of functions H corresponding to permutations of Q_k in accordance with the inclusions $m_j \in I_k$. Among these classes there is a single one allowing three relaxational curves which are periodical solutions of the degenerate system (5). This class of functions is characterized by $m_k \in I_k$ and through the rest of this section we consider functions H only of this class. Knowledge of the next few facts is instrumental in elaborations towards numerical experiments.

Lemma 1. Let x_μ be a continuous (in μ) periodical solution to equation (1) existing for $\mu \in (N, \infty)$, with $N > 0$ and let $h(\mu)$ be a Floquet multiplier for the variational equation corresponding to this solution. Then, $|h(\mu)| \to \infty$ when $\mu \to \infty$ and there exists $a \geqslant N$, such that for $\mu > a\, h(\mu)$ is of constant sign.

Proof. For a period of the periodic solution to the equations (1) or (3) there is an asymptotic expansion (as $\mu \to \infty$) received by the methods described in [14]. We need only the fact that the term, $c > 0$, is always present in the asymptotic expansion for the period $T(\mu)$ of the periodical solution to (3), regardless of the concrete type of the function H. Thus, the period, $T(\mu) = c +$ (lower order terms in μ^{-2}) [formula 7.8, Ch. 3 in 14]. (In the case of the Van der Pol equation, for example, $c = 1.614$.) Evaluation of Floquet multiplier $h(\mu)$ for periodical solution $x(t)$ to the equation (3) yields [9, page 169, equation 4.2.12]:

$$\lim_{\mu\to\infty} h(\mu) = \left[\lim_{\mu\to\infty} \frac{-\mu^2}{T(\mu)}\right] \lim_{\mu\to\infty} \int_0^{T(\mu)} \frac{dH}{dx}\,dt$$

$$= \lim_{\mu\to\infty} \frac{\mu^2}{T(m)} \lim_{\mu\to\infty} \left[\int_{x(0)}^{x(T(\mu)/2-0)} \frac{f^2(x)}{x}\,dx + \int_{x(T(\mu)/2+0)}^{x(T(\mu))} \frac{f^2(x)}{x}\,dx\right]$$

$$= \left[\int_{x_0}^{x_1} \frac{f^2(x)}{x}\,dx + \int_{x_2}^{x_3} \frac{f^2(x)}{x}\,dx\right]\left(\lim_{\mu\to\infty} \frac{\mu^2}{T(\mu)}\right). \tag{8}$$

In (8), x_0, x_1, x_2, x_3 are x-coordinates of end points on the pieces of the relaxational curve lying in the right and in the left half-planes. Since the number in brackets is different from zero (the sign of this number depends on a limit cycle) the statement of the lemma follows from (8).

Due to lemma 1, the dissipativity of (1), and the instability of the equilibrium (0, 0), the following facts hold:

(1) Out of the three limit cycles of equation (3) located in the neighborhoods of relaxational curves, the "intermediate" limit cycle is a repeller and the others are attractors.
(2) Equation (1) admits at least one limit cycle for any $\mu > 0$.

Recall that according to the previous notations,

$$H(r_1) = 0, \quad H(r_2) = 0, \quad 0 < r_1 < r_2.$$

The following lemma asserts that for a certain H, which allows for three relaxational curves, there are no bifurcational circles with radii in the segment $[0, r_2]$. Thus, taking into consideration the nature of the graph of H in Fig. 3 (which satisfies the conditions formulated in the beginning of this section), this means that there can be only one bifurcating circle for an oscillator with such an H.

The lemma also serves as an illustration of how to address technical details associated with synthesis for a variety of $kQsR$ types other than $1Q3R$ (to which the lemma is devoted).

Lemma 2. There exists a function H (of the class considered) such that equation (1) does not have bifurcating circles of radius $R \leqslant r_2$.

Proof. Consider a family Ω of smooth, strictly concave upward below the x-axis interval $[0, r_1]$ curves, with fixed left ends at (0, 0), fixed lowest points at the point (m_1, Q_1), $Q_1 < 0$

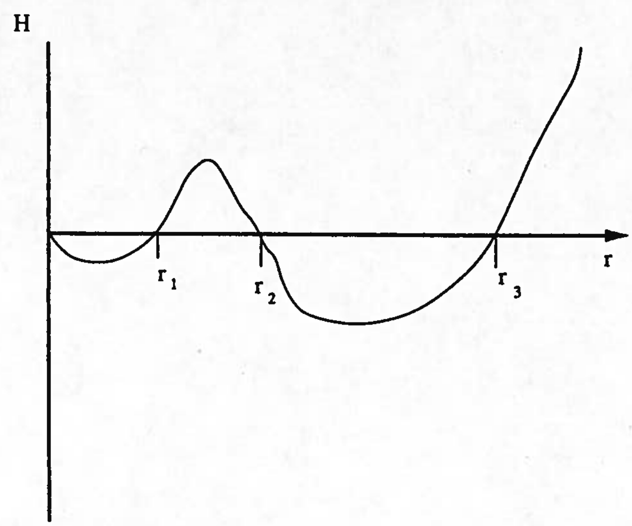

Fig. 3. Typical function $H(x)$ with 3 indicated positive roots, r_1, r_2, and r_3.

and each crossing the x-axis at the fixed point $(r_1, 0)$, $0 < m_1 < r_1$. Let Ω satisfy, additionally, the following conditions:

(a) The right end of each curve from Ω is located on the x-axis and has coordinates $(R, 0)$, where $r_1 < R$ and R depends on the curve.

(b) The curve from Ω is strictly concave downward above the x-axis interval $[r_1, R]$.

(c) Each curve from Ω has its highest point at the point with coordinates (m^*, Q_2) where Q_2 is fixed, $Q_2 > |Q_1|$ and $m^* \in (r_1, R)$, m^* depends on the curve.

Now, define the family, Ψ, of functions with the following properties:

(1) The graph of each function from Ψ coincides with one of the curves from Ω and each curve in Ω is the graph of a function from Ψ.

(2) The domain of each function from Ψ is the interval $[0, R]$, where R is the same as in condition (a) above.

Let K be a restriction on $[0, r_1]$ of some function from Ψ, and Ψ_1 is a subfamily of Ψ, $\Psi_1 \supseteq \Psi$, such that the restriction on $[0, r_1]$ of each function from Ψ_1 coincides with K.

Denote:

$$\gamma(R) = \cos^{-1}\left(\frac{r_1}{R}\right).$$

Resorting to (4) and introducing for each $q \in \Psi_1$

$$\prod\nolimits_q(R) := I_q(R) + J_q(R) = 4\int_0^{\gamma(R)} q(R\sin x)\sin^2 x\,dx + 4\int_{\gamma(R)}^{\pi/2} q(R\sin x)\sin^2 x\,dx \qquad (9)$$

note that,

$$\prod\nolimits_k(r_1) = 4\int_0^{\pi/2} K(r_1 \sin x)\sin^2 x\,dx = J(r_1) < 0.$$

Define for each $q \in \Psi_1$ with domain $[0, R_q]$ its extension q^* for $[0, \infty]$ by setting $q^*(t) = 0$ for all $t \geqslant R_q$. Then, due to the continuity in (9), there exists $0 < \varepsilon_1$, such that $\prod_q(R) < 0$ for $q \in \Psi_1$ with $\|K^* - q^*\|_L < \varepsilon_1$ and $R_q \leqslant r_2 = r_1 + \varepsilon_1$. This proves the lemma.

4. NUMERICAL ILLUSTRATIONS

We precede the description of the numerical investigations by the following remarks of a general nature. In Figs 4–7 the special types of dynamical systems to be subjected to detailed numerical analysis are demonstrated through their qualitative portraits without axes scales and labels. Functions H_j are shown only in graphical form and exclusively to emphasize the essential qualitative differences between them. Below will be given explicit realizations of H_j providing phase portraits for "small" and "large" μ.

In all examples, the construction of a proper function H is based on the procedure akin to that demonstrated in lemma 2. It is obvious, that the condition of concavity in lemma 2 is not essential and was used only for the sake of concreteness. For example, some of the desirable characteristics were "built" as linear aggregates of exponential functions and polynomials, this construction does not necessarily lead to non-linearities which are concave on the intervals between the roots. Note, that for similar purposes (obtaining the set of limit cycles) exponential non-linearities were proposed earlier in [7]. From lemma 2 follows:

Corollary. There exists an H such that equation (3) admits one bifurcation circle and equation (5) provides three relaxational curves. Such a $1Q3R$ case is presented in Fig. 4, and exact numerical results given in Fig. 1 with function $H = H_2$ written in Table 1 below.

Statements similar to that of the corollary for all other cases will be omitted.

Let $\{x_\mu\}$ be a family of periodical solutions continuous in μ in a neighborhood of $\mu = 0$. If new periodical solutions bifurcate from the family $\{x_\mu\}$, then Lyapunov exponent $h(\mu^*) = 0$

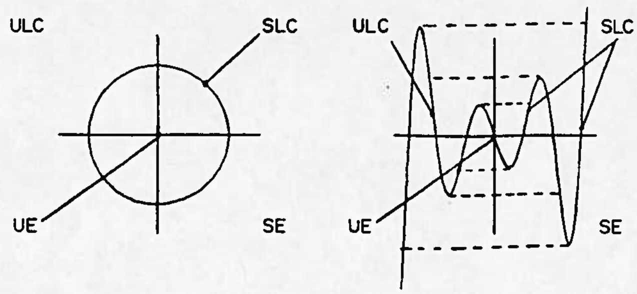

Fig. 4. $1Q3R$ oscillator.

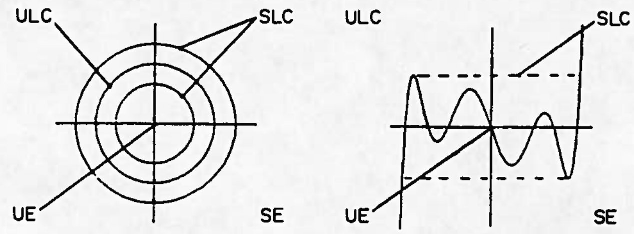

Fig. 5. $3Q1R$ oscillator.

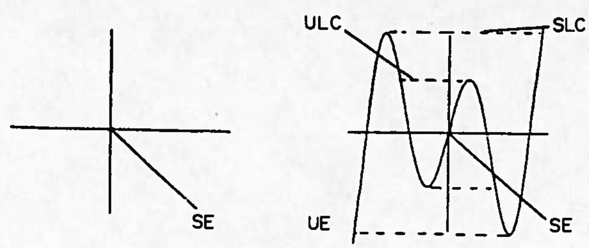

Fig. 6. $0Q2R$ oscillator.

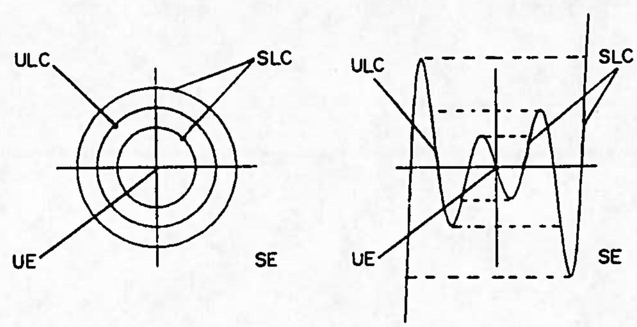

Fig. 7. $3Q3R$ oscillator.

Figs 4–7 show the qualitative types of phase portraits for the $1Q3R$, $3Q1R$, $0Q2R$ and $3Q3R$ oscillators, respectively. All equilibria and limit cycles are of the hyperbolic type. The symbols UE, SE, SLC, and ULC indicate unstable equilibrium, stable equilibrium, stable limit cycle, and unstable limit cycle, respectively. For each case, the phase plane portrait for "$\mu \to 0$" is on the left and the one for "$\mu \to \infty$" is on the right. Each picture on the right also presents the graph of the pertinent characteristic, H.

at the point of bifurcation. Therefore, the estimates of $h(\mu)$ served as indicators of the possibility for the existence (non-existence) of bifurcations. In all the ensuing examples, the Lyapunov exponent $h(\mu) \to 0$ as $\mu \to 0$ (quasilinear asymptotics) and according to lemma 1, $|h(\mu)| \to \infty$ when $\mu \to \infty$, if the above family exists on $(0, \infty)$. Since $|h(\mu)|$ turned out to be monotonically increasing in μ on any finite interval $(0, \mu)$ (clearly, this and similar statements should be understood in the sense of numerical empirical estimates), for each $\{x_\mu\}$

Table 1.

$$H_j(X) = \sum_k A_{jk} \exp((X - M_{jk})^2/\sigma_{jk}^2) + X^3$$

for $j = (1, 2)$.

$j = 1$						
$k =$	1	2	3	4	5	6
A_{1k}	2.20	− 1.00	1.00	− 2.20		
M_{1k}	− 0.90	− 0.30	0.30	0.90		
σ_{1k}^2	0.005	0.25	0.25	0.005		
$j = 2$						
A_{2k}	2.0	− 1.50	1.00	− 1.0	1.50	− 2.0
M_{2k}	− 0.80	− 0.70	− 0.30	0.30	0.70	0.80
σ_{2k}^2	0.001	0.0001	0.25	0.25	0.0001	0.001

and for $(j = 3, 4)$

$H_j(X) = \sum_k' C_{jk} X^k$ 'denotes sum over odd integers.

$j = 3$				
$k =$	7	5	3	1
C_{3k}	1.00000	− 21.06200	84.979166	− 65.41667
$j = 4$				
C_{4k}	0.58205200	− 2.429340	2.914700	− 0.985997

The number of significant figures shown for the numerical constants above is arbitrary but they reflect the precision involved in the intermediate calculations of the characteristics.

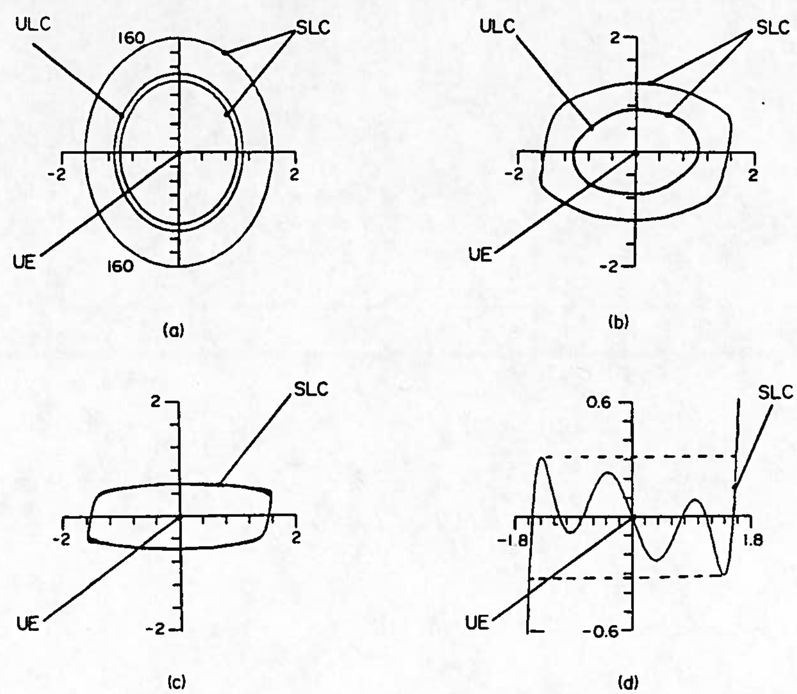

Fig. 8. Numerical phase portraits, dx/dt vs x, for $3Q1R$ with the characteristic H_4 of Table 1 (a) at $\mu = 0.01$, (b) at bifurcational value of $\mu = 0.422$, (c) at $\mu = 2.0$, (d) characteristic $H_4(x)$ vs x with indicated relaxational curves.

existing on $(0, \infty)$ and considered in our numerical investigations, no bifurcations from such $\{x_\mu\}$ exist (this is not to say that no bifurcations exist at all). Different results of numerical investigations (for all cases) pertinent to equation (3) with concrete H_j are represented in Figs 1, 2, and 8. In the case of H_3 there are three bifurcational circles and three relaxational curves (a $3Q3R$ class oscillator), however, no bifurcations exist at all. For this reason, this case is not paid further attention. Functions H_j are determined as the following.

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