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A PRACTICAL ALGORITHM FOR NUMERICAL DETERMINATION OF PERIODICAL REGIMES IN NONLINEAR OSCILLATORS

G.L. VIVIANI and Y.A. SAET
Lamar University, Beaumont, Texas 77710, USA

ABSTRACT: An effective practical algorithm based on array oriented computations, suitable for use with vector computational architectures, is described. The algorithm allows for deteremination of self-oscillations with unknown (in advance) periods, yet also includes the situation when the period is predetermined, particularly in the case of forced oscillations.

INTRODUCTION

The collocation method in nonlinear oscillations is described, demonstrated and broadly used in [1]. Even today the works [2, 3] (on forced oscillations) are still quoted for their concrete numerical results, regardless of significant gains in computational resources. Nevertheless, new and unexpected numerical solutions were found [4] even for the class of oscillators which have been considered in [2]. The Galerkin method used in [2], was not developed for autonomous systems although it was mentioned as a possibility in [5, p. 256] with no elaborated details. As for [1], in more than 150 pages of the text, only four pages are devoted to concrete numerical discussion of autonomous problems, and no viable algorithms are presented. Recently, [6] demonstrated accurate results only for the Van der Pol equation in the relaxational regime using ISL II software based on solving the Cauchy problem on a large temporal interval. However, such an approach becomes inefficient, or not possible, for oscillators with a number of different periodical regimes. Meanwhile, there is a growing necessity to address this problem due to new applications [7–9]. From time to time, different attempts of heuristic approaches appear (see for instance [10]) while a review of the area reveals persisting difficulties in numerical investigations of oscillations in autonomous systems.

As is known, the greatest difficulty with autonomous systems is that the period of the self-oscillation is unknown. In this work, the resolution of this obstacle, in a computationally feasible and efficient manner, is achieved.

LIENARD OSCILLATOR

For concreteness, the algorithm is demonstrated with the example of a strongly nonlinear Lienard Equation:

$$\ddot{x} + \mu Q(x)\dot{x} + G(x) = R(t), \quad \mu > 0 \tag{1}$$

where $Q(x)$ and $G(x)$ are polynomials of arbitrary degrees and $G(x)$ is such that $G(x)x > 0$ for $x \neq 0$, $G(0) = 0$. (For the autonomous oscillator, $R(t) \equiv 0$). Such an oscillator is represented by a circuit shown in Fig. 1, (in this case $G(x) \equiv x$, with obvious time rescaling), where $H(v)$ is a voltage dependent nonlinear 1-port. In eq. (1), $Q(x) = \mathrm{d}H(x)/\mathrm{d}x$. In Fig. 1 the current-voltage characteristic, $H(v)$, corresponds to the data in Example 2 below. Dotted lines show the one stable relaxational limit cycle. Such characteristics are realizable and have presently found some applications. If $x(t)$ is a periodic solution to eq. (1), with period T, then it can be represented by its Fourier Series,

$$x(t) = \sum_{k}^{\infty} (a_k \cos kwt + b_k \sin kwt)$$

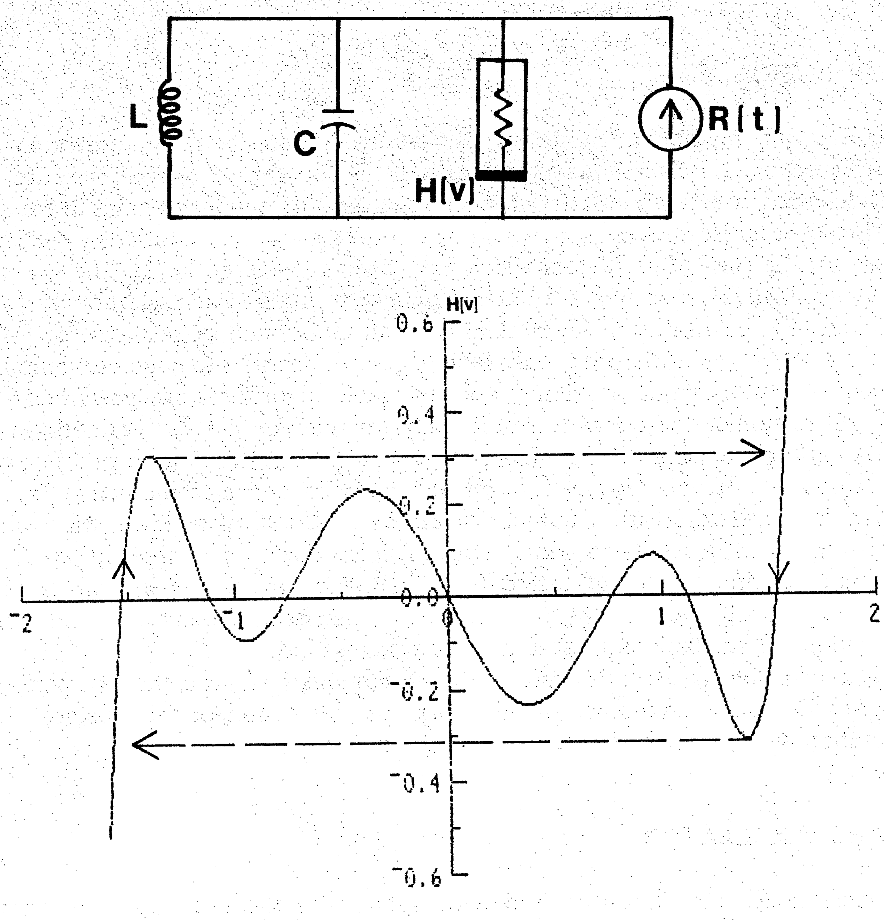

Figure 1. Nonlinear oscillator circuit.

with

$$w \equiv \frac{2\pi}{T}$$

We introduce the following necessary notations,

$$x_M(t) = \sum_{k=0}^{M} (a_k \cos kwt + b_k \sin kwt) \tag{2}$$

with positive integer M, vector X^Γ,

$$X^\Gamma \equiv (a_1, b_1, a_2, b_2, \ldots, a_M, b_M)^*$$

vector X^P,

$$X_P \equiv (X_M(t_0), X_M(t_1), \ldots, X_M(t_p))^*$$

where

$$t_k = k\frac{T}{(P+1)} \quad k = 0, 1, 2, \ldots, P \tag{3}$$

and $P \geqslant 2M$. (The symbol * denotes transposition.)

There exists [1, p. 124] a matrix, Ω, with P rows and $2M$ columns such that,

$$X^P = \Omega X^\Gamma \tag{4}$$

which is uniquely determined, by identical equality with respect to t_k's, in eq. (3). In the same way there exists a unique matrix, Γ, with 2 M rows and P columns such that,

$$X^\Gamma = \Gamma X^P . \tag{5}$$

Denoting the sth order derivative,

$$X^P_{(s)} \equiv \frac{d^s}{dt^s} X^P(t),$$

it is straightforward to calculate a matrix W^s, such that,

$$X^P_{(s)} = W^s X^P = W^s X^P_{(0)} . \tag{6}$$

ALGORITHM

We will first describe an algorithm for forced oscillations, from which the algorithm for autonomous systems is developed.

Forced Oscillations

With T known, and $P = 2M$, the development of the collocational scheme for eq. (1) yields,

$$Z(X^P) = W^2X^P + \mu Q^P(X^P)W^1X^P + G^P - R^P = 0 \tag{7}$$

where

$$Q^P(X^P) = (c_N(X^P)^N + c_{N-1}(X^P)^{N-1} + \cdots + c_0 e)$$

with $(X^P)^{N-k}$ denoting a vector that is componentwise raised to the $(N-k)$th power where N is the order of the polynomial $Q(x)$ with coefficients c_k, $(k = 0, 1, \ldots, N)$, and e is a column vector with number 1's as its entries, of the same dimension as X^P. G^P and R^P are received in the same manner as Q^P and X^P,

$$G^P(X^P) = (g_N(X^P)^N + g_{N-1}(X^P)^{N-1} + \cdots + g_0 e)$$

with coefficients, g_k, and,

$$R^P \equiv (R(t_0), R(t_1), \ldots, R(t_p))^*.$$

Introducing ${}^0X^P$ as an initial guessed solution, we define,

$${}^1X^P = {}^0X^P - \propto \left(\frac{\partial Z}{\partial\, {}^0X^P}\right)^{-1} Z({}^0X^P) \tag{8}$$

where, the k in ${}^kX^P$ denotes the kth iteration, and $\propto$ is an externally controlled parameter for adjusting convergence characteristics. We obtain,

$$\frac{\partial Z}{\partial X^P} = W^2 + \mu\left(\frac{\partial Q^P}{\partial X^P}\right)W^1X^P + \mu Q^P W^1 + \left(\frac{\partial G^P}{\partial X^P}\right) \tag{9}$$

REMARK 1: The above scheme, relying on eqs. (9) and (8), allows for efficient iterations, especially when one notes that $W^{(i)}$ remains fixed (see Appendix). Contrary to the algorithms in [2], our formulas are presented in array organized computations. It should also be noticed that the algorithm described is a further development of the collocational scheme in [1].

Consideration of Self-Oscillations

For the autonomous equation, $R(t) \equiv 0$, and, T is a new unknown parameter. Changes to the numerical scheme described for forced oscillations result in the following:

After introducing $P = 2M + 1$ equidistant collocational points (instead of $2M$), the matrix of eq (4), Ω, now has $2M + 1$ rows and $2M$ columns, as does the matrix W^s of eq. (6).

As a result, the dimension of Z is greater by one, (compared to the nonautonomous equation) and the vector of unknowns is redefined to be $V^P = (X^P, T)^*$. By application of the chain rule to the equation for Z,

$$Z(X^P) = W^2X^P + \mu Q^P(X^P)W^1X^P + G^P = 0 \tag{10}$$

one obtains

$$\frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{V}^P} = \frac{\partial(\boldsymbol{X}^P\ 0)}{\partial \boldsymbol{V}} \frac{\partial \boldsymbol{Z}}{\partial(\boldsymbol{X}^P, 0)} + \frac{\partial(0, T)}{\partial \boldsymbol{V}} \frac{\partial \boldsymbol{Z}}{\partial(0, T)} \tag{11}$$

where the vector $(0, T)$ has P zero elements and the vector $(\boldsymbol{X}^P, 0)$ has one zero element and hence both have the same dimension as vector $\boldsymbol{V}$. The result is that each term of the right hand side of eq. (11) is multiplied by a diagonal matrix. The revised iterative formula for $\boldsymbol{V}$ becomes.

$${}^1\boldsymbol{V}^P = {}^0\boldsymbol{V}^P - \propto \left(\frac{\partial \boldsymbol{Z}}{\partial {}^0\boldsymbol{V}^P}\right)^{-1} \boldsymbol{Z}({}^0\boldsymbol{V}^P) \tag{12}$$

The following formula is used for calculating $\boldsymbol{W}^s$'s as,

$${}^{i+1}\boldsymbol{W}^s = \frac{i+1w}{iw}\ {}^i\boldsymbol{W}^s\,.$$

A similar calculation is used for revising elements of the matrix of the second term in eq. (11), which involves derivatives of $\boldsymbol{W}^s$.

REMARK 2: It should be noted that finding the period is an inalienable part of the algorithm. Also, the convergence characteristics of the Newton-Kantarovich iterations of eqs. (8) and (12) benefited from a subiteration (with respect to $\propto$) to insure reduction in the norm of the error.

EXAMPLES

Analytical results on the existence of a family of limit cycles for the Lienard Equation can be found in [11, 12]. For "very large" $\mu \gg 1$, the self-oscillations can be found on the basis of geometrical constructions in the phase plane using the nonlinear current-voltage characteristics of the 1-port. The description of such relaxational limit cycles on the phase plane does not involve the differential equation itself [13, Ch. 3], but finding the period takes serious efforts, see [13] and [14]. As we shall see in examples, methods of solution for large μ, do not allow one to determine all limit cycles for intermediate μ.

This algorithm allows for easy determination of more than one periodical solution. In [7], for this purpose, the analytical method of small parameter was used. However, the numerical effectiveness of the method is very limited. The same problem exists even for nonautonomous systems. For instance, in the examples of forced oscillations considered in [2], only one periodical solution was found, while it is known that there are several of them for a specific set of parameters [15].

The proposed algorithm was implemented on a mainframe computer. For illustrative purposes, certain concrete characteristics of the nonlinear element in the circuit were chosen. Specifically, $Q(x)$ was assumed to be an even function and $G(x)$ was assumed to be odd, assuring that the solution $x(t)$, would have the symmetry property,

$$x(t) = -x(t + T/2) \quad \forall t$$

This allowed for the points t_k, to be equidistant on the interval $[0, T/2]$ (instead of $[0, T]$), thus reducing to $P = M + 1$, the total number of points considered in eq. (3). Due to the symmetry, elements of $\boldsymbol{X}^\Gamma$, with even numbers, vanish. The formula's for $\boldsymbol{W}^s$ are given in the Appendix. Due to memory allocation restrictions, P was limited to $P \leq 231$.

For the following problems a truncated solution vector of coefficients, $\boldsymbol{X}^\Gamma$, is displayed in two columns, with each row corresponding to the pair, (a_k, b_k) $k = 1, 3, 5, \ldots$ Those values which are not indicated are neglected.

EXAMPLE 1: The Van der Pol Oscillator, is chosen for comparison with previous results. Due to the above symmetry property, the formulas of the Appendix are applicable. For this case, $M = 231$, $\mu = 5$, $c_2 = 1$, $c_1 = 0, c_0 = -1$, $g_1 = 1$, $g_0 = 0$ (see formulas for $Q(\boldsymbol{X})$ and $G(\boldsymbol{X})$). The solution for $\boldsymbol{X}^\Gamma$ and T, with the magnitude of the maximum component in vector $\boldsymbol{Z}$ less than 10^{-4}, is,

$\boldsymbol{X}^\Gamma =$

0.63975	2.00656
0.193832	0.550989
0.116427	0.283401
0.0800519	0.174215
0.0578738	0.116588
0.0428625	0.0819679
0.0321731	0.0534932
0.0243470	0.0441395
0.0185223	0.0332704
0.0185223	0.0332704
0.0141419	0.0253746
0.0108249	0.0195272
0.00830130	0.0151327
0.00637472	0.0117925
0.00490021	0.00923065
0.00376954	0.00725180
0.00290124	0.00571441
0.00223368	0.00451435
0.00171999	0.00357395
0.00132443	0.00283460
0.00101970	0.00225174

$T = (1.85)2\pi$.

EXAMPLE 2: Here $G(x) \equiv x$, and $Q(x)$ is determined as a polynomial with $c_6 = 4.07437$, $c_4 = -12.1467$, $c_2 = 8.7441$, $c_0 = -0.985997$, $c_k = 0$, $k = 1, 3, 5$, $M = 231$, $\mu = 1.2$. Three solutions were determined, indicating two stable and one unstable limit cycle as shown in the phase plane plot of Fig. 2.

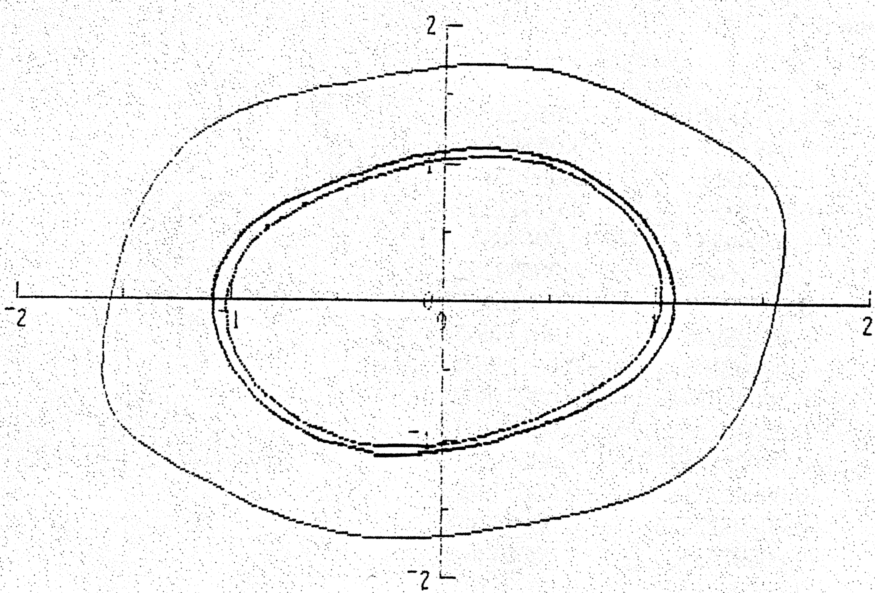

Figure 2. Phase plane representation of limit cycles for Example 2.

Solution 1

$\boldsymbol{X}^{\Gamma}=$

8.76049E-2	1.01149E0
−6.85327E-2	1.71688E-2
−2.07840E-2	8.61603E-3
−4.39407E-3	−3.09938E-3
−1.05242E-3	−1.50743E-3
8.58229E-5	−5.55174E-4

$T=(1.02562)2\pi$

with the magnitude of the maximum component in vector $\boldsymbol{Z}$ less than 10^{-6}.

Solution 2

$\boldsymbol{X}^{\Gamma}=$

7.52977E-2	1.07398E0
−5.88986E-2	8.53827E-3
−2.58466E-2	1.00207E-2
−4.00712E-3	−2.60972E-3
−1.29606E-3	−1.90544E-3
2.82830E-5	−5.64474E-4
1.51327E-4	−1.97558E-4

$T=(1.02057)2\pi$

with the magnitude of the maximum component in vector $\boldsymbol{Z}$ less than 10^{-6}.

Solution 3

$X^\Gamma =$

0.0611846	1.65626
−0.0487357	0.0664727
0.00205155	−0.039654
−0.0102421	0.0141221
−0.0184955	0.0065687
−0.00116644	−0.00589262
0.00389732	−0.000169571
−0.00128906	0.00114667
−0.00102106	−0.00123294
0.000795964	−0.000318091
0.000104261	0.000521417
−0.000287257	−0.0000897117
0.000111618	−0.000178804
0.0000822526	0.000102499
−0.0000753523	0.000033612
−0.0000021451	−0.000051067

$T = (1.02622)2\pi$

with the magnitude of the maximum component in vector $\boldsymbol{Z}$ less than 10^{-6}.

REMARK 3: The examples shown are intended to verify the method. In Example 2, note the close locations of two of the limit cycles in the phase plane. It is very difficult to distinguish between these two limit cycles with certainty, using a Cauchy–Ėuler like method. The asymptotic method [13], as used in Fig. 1, is also limited in that it does not reveal the existence of these solutions, while our algorithm clearly reveals three different solutions.

APPENDIX

The following expressions apply for the symmetry,

$$x(t) = -x(t + T/2)$$

for the autonomous Lienard oscillator. Odd symmetry is guaranteed if in eq. (1) $G(x)$ is an odd function and $Q(x)$ is an even function. $P = M + 1$.

Defining the vector $\boldsymbol{X}^\Gamma$,

$$\boldsymbol{X}^\Gamma \equiv (a_1, b_1, a_3, b_3, \ldots, a_M, b_M)^*$$

and letting $\boldsymbol{X}^P$ be

$$\boldsymbol{X}^P \equiv (x_M(t_0), x_M(t_1), \ldots, x_M(t_P))^*$$

with equidistant points chosen on the interval $[0, T/2]$,

$$t_k = k\frac{T}{(P+1)} \quad k = 0, 1, 2, \ldots P+1 . \tag{A.1}$$

Letting

$$w \equiv \frac{2\pi}{T}$$

then the elements of the rectangular matrix W^j, are

$$W^j_{pk} = \frac{2w^j}{M+1} \sum_{s=1}^{M}{}' (-1)^h s^j \cos\left[(p-1)s\frac{\pi}{M+2} - (k-1)s\frac{\pi}{M+1}\right]$$

when $j = 2h$ (A.2)

$$W^j_{pk} = \frac{2w^j}{M+1} \sum_{s=1}^{M}{}' (-1)^h s^j \sin\left[(k-1)s\frac{\pi}{M+1} - (p-1)s\frac{\pi}{M+2}\right]$$

when $j = 2h+1$

$$k = 1, 2, \ldots, M+1; \quad p = 1, 2, \ldots, M+2; \quad h = 0, 1. 2, \ldots,$$

and Σ' denotes summation over odd subscripts. There is a more simplified expression for W^0,

$$W^0_{pk} = \frac{2}{M+1} \sum_{s=1}^{(M+1)/2}{}' \cos\left[(2s-1)\pi\left(\frac{p-1}{M+2} - \frac{k-1}{M+1}\right)\right]$$

$$k = 1, 2, \ldots, M+1; \quad p = 1, 2, \ldots, M+2$$

is available.

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